Strategic Randomization

Equilibria in Markovian Stopping Games

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joint work with Boy Schultz and Kristoffer Lindensjö

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Reminder: Game Theory and Optimal Stopping

Two player normal form games

Player	1	2
Strategy sets	$\mathcal{S}^1$	$\mathcal{S}^2$
Strategies	$s_1 \in \mathcal{S}^1$	$s_2 \in \mathcal{S}^2$
Rewards	$J^1 : \mathcal{S}^1 \times \mathcal{S}^2 \to \mathbb{R}$	$J^2 : \mathcal{S}^1 \times \mathcal{S}^2 \to \mathbb{R}$
Goal	maximize J^1 over $\mathcal{S}^1$	maximize J^2 over $\mathcal{S}^2$

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$(s_1^*, s_2^*) \in \mathcal{S}^1 \times \mathcal{S}^2$ is called Nash equilibrium, if

$$J^1(s_1^*, s_2^*) \geqslant J^1(s_1, s_2^*)$$
$$J^2(s_1^*, s_2^*) \geqslant J^2(s_1^*, s_2)$$

for all $(s_1, s_2) \in \mathcal{S}^1 \times \mathcal{S}^2$.

Example: Odd vs. Even

- $\mathcal{S}^1 := \mathcal{S}^2 := \{\text{Odd}, \text{Even}\}$
- J^1, J^2 given by ...

s_1 / s_2	Odd	Even
Odd	(1,-1)	(-1,1)
Even	(-1,1)	(1,-1)

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- No Nash equilibrium

Randomized equilibria

- Enlarge the spaces of strategies from e.g. $\mathcal{S}^1 = \mathcal{S}^2 = \{\text{Odd, Even}\}$ to the space

$$\mathcal{M}^1(\{\text{Odd, Even}\}) := \{P : P \text{ is a probability measure on } \{\text{Odd, Even}\}\}.$$

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- Extend J^1, J^2 to $\mathcal{M}^1(\{\text{Odd, Even}\})^2$ via

$$J^i(P_1, P_2) := \int\int J^i(s_1, s_2) P_1(ds_1) P_2(ds_2), \quad i = 1, 2.$$

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- $(P^*, P^*) \in \mathcal{M}^1(\{\text{Odd, Even}\})^2$, $P^*(Odd) = P^*(Even) = \frac{1}{2}$ is a randomized Nash equilibrium in the Odd vs. Even game.

Randomization

1st Observation

Randomization is necessary for general existence of Nash equilibria.

Markovian Optimal Stopping Problems

X nice Markov processes,

$$V(x) = \sup_{\tau} E_x(e^{-r\tau} g(X_\tau))$$

Optimal stopping time

Under weak assumptions, the following first exit time is optimal:

$$\tau^* = \inf\{t : X_t \notin C\}, \quad C = \{x : V(x) > g(x)\}$$

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- allows for explicit solutions (PDE-formulations,...)
- time-consistent optimizer (subgame perfection)

Markovian optimal stopping

2nd Observation

For both practical and interpretative purposes, one should look at Markovian stopping times.

Dynkin games

Example: Dynkin games

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$$(x, \tau_1, \tau_2) \mapsto \mathbb{E}_x[\mathbb{1}_{\tau_1 \leqslant \tau_2} e^{-r\tau_1} g_1(X_{\tau_1}) + \mathbb{1}_{\tau_1 > \tau_2} e^{-r\tau_2} f_1(X_{\tau_2})],$$
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Important special cases:

- $g_1 = -f_2, \; g_2 = -f_1$: zero-sum
- $g_1 \leqslant f_1, g_2 \leqslant f_2$: war of attrition
- ...

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(τ_1^*, τ_2^*) is called **Nash equilibrium** if

$$J^1(x, \tau_1^*, \tau_2^*) \geqslant J^1(x, \tau_1, \tau_2^*),$$
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for all $(\tau_1, \tau_2) \in \mathcal{T}^2$.

Equilibria in Dynkin games

There are two types of existence results for equilibria

structural assumptions	restrictive	rather general
equilibrium strategy	first exit time	randomized stop. time
Subgame perfect (Markovian)	Yes	No

Markovian Stopping Games

Want a class of stopping times that

- is large enough for existence of equilibria
- is manageable and interpretable
- respects Markovian framework (subgame perfection)

Randomized Markovian stopping times

Stopping with expectation constraint, Pre-commitment

$$V(x) = \sup_{\tau} E_x(g(X_\tau)), \quad \text{sj.t. } E_x(\tau) \leqslant T$$

- S. Ankirchner, M. Klein et al (2019, 2019), E. Bayraktar et al (2020, 2022)

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- X_{τ^*} just takes three values
- One possibility: $a \leqslant c \leqslant b$, $E \sim \text{Exp}(1)$ independent from X:

$$\tau^{[a,b],\lambda\delta_c} := \inf\{t \geqslant 0 : X_t \notin [a,b] \text{ or } \lambda L_t^c \geqslant E\}$$

Stopping with expectation constraint, time-consistent (C., Klein,Schultz, AMO, 2025)

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- Equilibrium stopping time (in the sense of Strotz)?
- $R \subseteq C$, $E \sim \text{Exp}(1)$ independent of X:

$$\tau^{C,\frac{1}{T}\text{Leb}_R} := \inf\left\{t \geqslant 0 : X_t \notin C \text{ or } \frac{1}{T}\int_0^t 1_{X_s \in R} ds \geqslant E\right\}$$

Non-standard stopping problems 1, time-consistent (C., Lindensjö, SPA, 2020)

$$\sup_{\tau} E_x(f(X_\tau)) + g(E_x(h(X_\tau))),$$

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- C, ψ function, $E \sim \mathrm{Exp}(1)$ independent of X:

$$\tau^{C,\psi(x)dx} := \inf\left\{t \geqslant 0 : X_t \notin C \text{ or } \int_0^t \psi(X_s)ds \geqslant E\right\}$$

Non-standard stopping problems 2, time-consistent (Bodnariu, C., Lindensjö, SICON, 2024)

$$\sup_{\tau} E_x\left(h(\tau)g(X_\tau) + \int_0^\tau h(s)f(X_s)ds\right),$$

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Dynkin games with heterogeneous beliefs (Ekström, Glover, Leniec, JAP, 2017)

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Summary

- general existence theorems only in general randomised stopping times: too large, no subgame-perfection
- equilibria in first exit times just in special problem classes
- In continuous time, stopping times of the form

$$\inf\left\{t \geqslant 0 : X_t \notin C \text{ or } \int_0^t \psi(X_s)ds + \sum_i d_i L_t^{x_i} \geqslant E\right\}$$

in some examples via guess-and-verify.

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$$\tau^{C,\lambda} := \inf\{t \geqslant 0 : X_t \notin C \text{ or } A_t^\lambda \geqslant E\}.$$

- space of randomized Markovian times

$$\mathcal{R} := \{\tau^{C,\lambda} : C \text{ open}, \lambda \in \mathrm{RM}(C)\}$$

Markovian?

- For $\tau^{C,\lambda} \in \mathcal{R}$ we set

$$\theta_{\tau^B} \circ \tau = \inf\{t \geqslant 0 : \theta_{\tau^B} \circ A_t^{C,\lambda} \geqslant E'\}$$

with $E' \sim \mathrm{Exp}(1)$ independent from X, E.

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with $E' \sim \mathrm{Exp}(1)$ independent from X, E.

- This yields a strong Markov property for $\tau^{C,\lambda}$:

$$\mathbb{1}_{\tau^{C,\lambda} \geqslant \tau^B} \tau^{C,\lambda} \stackrel{\mathrm{d}}{=} \mathbb{1}_{\tau^{C,\lambda} \geqslant \tau^B} (\theta_{\tau^B} \circ \tau^{C,\lambda} + \tau^B)$$

Equilibria in Dynkin games

Can we get the best of both worlds?

structural assumptions	restrictive	fairly general	fairly general
equilibrium strategy	first exit time	**randomized Markovian time**	measure on $\mathcal{T}$
Subgame perfect (Markovian)	Yes	Yes	No

$$\mathcal{R}: \text{ set of all } \tau^{C,\lambda} := \inf\{t \geqslant 0 : X_t \notin C \text{ or } A^\lambda_t \geqslant E\}.$$

Randomized Markovian stopping times suitable class?

Main question

In general Markovian stopping games

- do equilibria exist
- can equilibria be constructed explicitly

in the class of randomized Markovian stopping times?

Construction: zero-sum

Zero-Sum

$$J(x;\tau_1,\tau_2) := \mathbb{E}_x\left(e^{-r(\tau_1\wedge\tau_2)}\left(f(X_{\tau_1})\mathbb{I}\{\tau_1<\tau_2\}+g(X_{\tau_2})\mathbb{I}\{\tau_1>\tau_2\}+h(X_{\tau_1})\mathbb{I}\{\tau_1=\tau_2\}\right)\right)$$

General Result under ordering condition (Ekström, Peskir, 2008, 2009...)

If $f \leqslant h \leqslant g$, an equilibrium can be found in the class of first exit times using the semiharmonic characterization.

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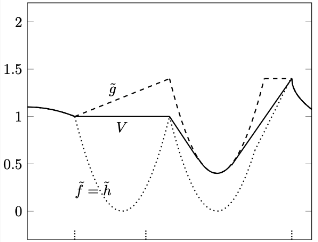

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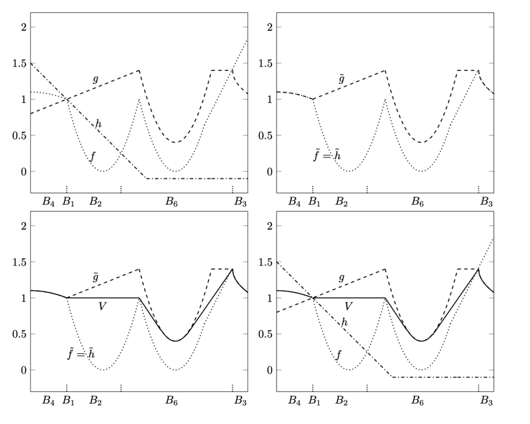


General Result (C., Lindensjö, 2024)

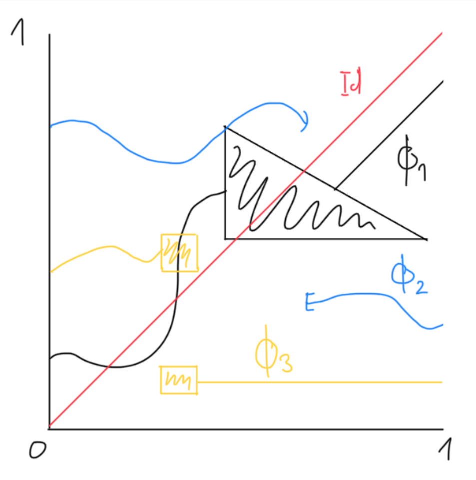
Without any ordering condition, in the zero sum game global Markovian randomized ϵ-Nash equilibria can be constructed for each $\epsilon > 0$.

Existence of Markovian equilibria in war-of-attrition-games

Fixed point theorems

Assumptions in your typical fixed point theorem:

- Compact (pre-)image.
- *Some* continuity (a closed graph) of the mapping.
- Convexity (or a variant) and non-emptyness of the image sets of the mapping.



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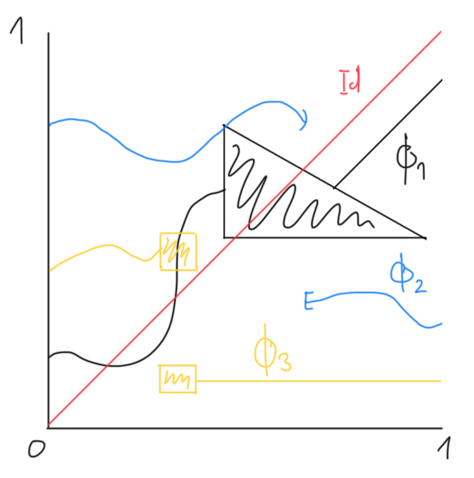
- Solving a fixed point problem means to find a topology with these properties.
- $\mathfrak{R}$ has no canonical topology.



Topologies on $\mathcal{R}$

We embed $\mathcal{R}$ into another space via ι and equip it with the pullback topology.

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- $\iota_x : \mathcal{R} \to \mathcal{M}^1([0,\infty] \times I) := \{\mu | \mu \text{ probability measure on } [0,\infty] \times I\}$

$$\tau^{C,\lambda} \mapsto P_x^{(\tau^{C,\lambda}, X_{\tau^{C,\lambda}})}.$$

Topologies on $\mathcal{R}$ cont'd

Properties	$\mathfrak{T}^{\iota_1}$	$\mathfrak{T}^{\iota_2}$	$\mathfrak{T}^{\iota_x}$
Space	measurable functions	measures	probability measures
Compactness	unlikely	only for finite I	yes, with tightness
Φ continuity	more likely	under some conditions	under some conditions
Convexity	?	yes	In some sense...

Markovian Dynkin games, diffusion

- $J_1(x, \tau_1, \tau_2) = \mathbb{E}_x[\mathbb{1}_{\tau_1 \leqslant \tau_2} e^{-r\tau_1} g_1(X_{\tau_1}) + \mathbb{1}_{\tau_1 > \tau_2} e^{-r\tau_2} f_1(X_{\tau_2})]$

Class of Markovian stopping times $\mathcal{R}$ given by C, λ :

$$\tau^{C,\lambda} := \inf\{t \geqslant 0 : A_t^{C,\lambda} \geqslant E\}, E \sim \text{Exp}(1).$$

Existence Theorem (C., Schultz, 2024, see also Decamps, Gensbittel, Mariotti)

Assume $g_1 \leqslant f_1, g_2 \leqslant f_2$, then (under weak assumptions) there exists a Nash equilibrium $(\tau^{C^{(1)},\lambda^{(1)}}, \tau^{C^{(2)},\lambda^{(2)}})$ in $\mathcal{R}$.

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3. Use Kakutani's theorem with the topology $\mathfrak{T}^{\iota_2}$ to find equilibria in the auxiliary games for each n.
4. Use compactness of $\bigtimes_{x \in \mathbb{Q}} \mathfrak{T}^{\iota_x}$ to find a limit point of the sequence of the equilibria of the auxiliary games.
5. By continuity of J^1, J^2 the limit point is an equilibrium of the original game.

Conclusion

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- Can be used for direct constructions
- Topology on $\mathcal{R}$ crucial for general existence

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- Topology on $\mathcal{R}$ crucial for general existence
- Recent application of $\mathcal{R}$ in Diffusion Generative Models (ML)