Worst-Case Optimal Investment in Incomplete Markets

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Worst-Case Optimal Investment in a Nutshell

The Market Model

- Usual Black-Scholes model:

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db_t &= b_t r dt, \quad b(0) = 1 \\
dS_t &= S_t \left[(\lambda + r) dt + \sigma dW_t \right], \quad S_0 = s
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with constant market coefficients λ and $\sigma \neq 0$.

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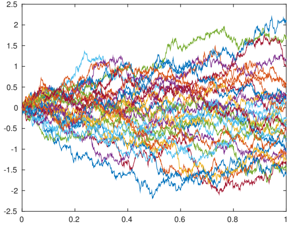
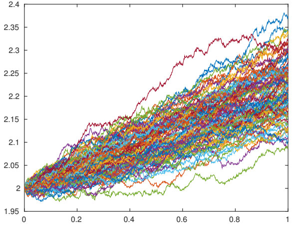


Some paths of W



Some scenarios for S

Worst-Case Optimal Investment

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- In general: Finitely many crashes can happen before the horizon T.
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- Studied for the first time in Korn & Wilmott (2002).

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$$\begin{aligned}
\frac{dX_t}{X_t} &= (r + \pi_t \lambda)dt + \pi_t \sigma dW_t \text{ on } [0, \tau), \quad X_0 = x \\
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- $(\tilde{X}_t^\pi)_{t\in[0,T]}$: wealth process in the standard crash-free Black-Scholes model corresponding to the portfolio process π.
- Explicit expression for $\tilde{X}^\pi$:

$$\tilde{X}_t = x\exp\left(\int_0^t\left(r+\pi_s\lambda-\frac{1}{2}\pi_s^2\sigma_s^2\right)ds+\int_0^t\pi_s\sigma_s dW_s\right).$$

Worst-Case Optimal Investment Problem

- The problem to optimally choose a pre- and post-crash strategy $(\pi, \overline{\pi}) \in \mathcal{A}(t,x) \times \overline{\mathcal{A}}(t,x)$ facing the worst possible crash-scenario τ with $0 \leq \tau \leq T$, i.e.

$$\sup_{(\pi,\overline{\pi})} \inf_{\tau} \mathbb{E}\left[U(X_T^{\pi,\overline{\pi}})\right] \tag{P}$$

with final wealth $X_T^{\pi,\overline{\pi}}$ in the case of a crash of size ℓ at τ given by

$$X_T^{\pi,\overline{\pi}} = (1 - \pi_\tau \ell)\, \tilde{X}_T^{\pi}$$

is called the **worst-case portfolio problem**.

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COM Device - Merton Problem with Random Initial Time
We can solve for X explicitly (using e.g. power utility $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$)

$$U(X_T^{\overline{\pi}}) = U(X_\tau^{\overline{\pi}}) \exp\left((1-\gamma)\int_\tau^T \Phi(\overline{\pi}_s) ds\right) M_T(\overline{\pi})$$

with $X_\tau^{\overline{\pi}} = (1-\pi_\tau \ell) X_\tau^{\pi}$, a martingale $M(\pi)$ satisfying $M_\tau(\pi) = 1$ and

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COM Device - Merton Problem with Random Initial Time for LOG-utility

$$U(X_T^{\overline{\pi}}) = U(X_\tau^{\overline{\pi}}) + \int_\tau^T \Phi(\overline{\pi}_s)ds + M_T(\overline{\pi})$$

with $X_\tau^{\overline{\pi}} = (1-\pi_\tau\ell)X_\tau^{\pi}$, a martingale $M(\pi)$ satisfying $M_\tau(\pi) = 0$ and

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Worst-case problem (P) decouples into the post-crash problem (P_{post}) and the problem to choose a pre-crash strategy such that

$$\sup_{\pi} \inf_{(\tau,\ell)} \mathbb{E}\left[\overline{V}(\tau,(1-\pi_\tau \ell)X^\pi_\tau)\right] \qquad (\mathrm{P}_{pre})$$

where $\overline{V}$ denotes the value function of the post-crash (Merton) problem:

$$\overline{V}(t,x) = \frac{x^{1-\gamma}}{1-\gamma} e^{\left((1-\gamma)\int_t^T \Phi(\bar{\pi})\,ds\right)} = U(x) e^{\left((1-\gamma)\int_t^T \Phi(\bar{\pi})\,ds\right)}.$$

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Controller-vs-Stopper Game

- (P_{pre}) is a controller-vs-stopper game and Seifried (2010) has shown that this is solved by rendering

  $$t \mapsto \overline{V}\left(t, (1-\pi_t \ell) X_t^\pi\right)$$

  a continuous martingale, since then the market's (stopper's) actions become irrelevant to the investor (controller).
- Apply Itô's formula to $\overline{V}$: $\Rightarrow$ WOC-ODE.

Solutions to the WOC Problem

- Optimal PRE-CRASH strategy: Unique solution of the ODE

$$\pi_t' = \frac{1 - \pi_t \ell}{\ell} \left[-\frac{\gamma \sigma^2}{2} \left(\pi_t - \pi^M \right)^2 \right] \ , \ \pi_T = 0 \, .$$

[$\Rightarrow$ $\bar{V}$ is a martingale **Argument/reason behind:** An investor has to be indifferent between a crash happening immediately or not at all.]

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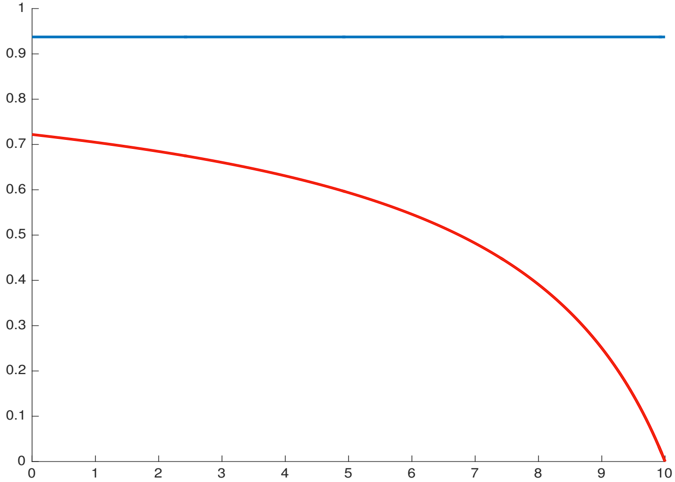
- Optimal POST-CRASH strategy: Merton fraction $\pi^M = \lambda/\gamma\sigma^2$.
- Log: Explicit calculations as given in Korn & Wilmott (2002).
- Power: Solution of HJB systems as in Korn & Steffensen (2007) or using the martingale approach of Seifried (2010).

Illustration: $\hat{\pi}$ (red) and π_M (blue)



Parameters: $\gamma = 1$, $\lambda = 0.15$, $\sigma = 0.4$, $\ell = 0.2$, $T = 10$

Stochastic Lévy Market Coefficients

Choose pre-crash and post-crash strategy $(\pi, \overline{\pi}) \in \mathcal{A}(t,x) \times \overline{\mathcal{A}(t,x)}$ as to maximize the LOG-utility of terminal wealth in the worst-case scenario:

$$\sup_{(\pi,\overline{\pi})} \inf_{\tau} \mathbb{E}[\log X_T^{\pi,\overline{\pi}}]. \qquad (\mathrm{P}^{SM})$$

Now, $X^{\pi,\overline{\pi}}$ is the solution X to

$$\frac{dX_t}{X_{t-}} = (r_t + \pi_t \lambda_t)dt + \pi_t \sigma_t dW_t - \int_{[0,l^{\max}]} \pi_t l \nu(dt, dl) \quad \text{on } [0,\tau)$$

$$X_\tau = (1 - \pi_\tau \ell) X_{\tau-}$$

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and initial condition $X_0 = x > 0$, where ν is a Poisson random measure with Lévy measure ϑ with $l^{\max} \ll \ell$.

Analogous to the constant case, we define the function

$$\Phi_t : [0,\infty) \to \mathbb{R}^\Omega, y \mapsto r_t + \lambda_t y - \frac{1}{2}\sigma_t^2 y^2 - \int_{[0,l^{\max}]} \log(1 - yl)\vartheta(dl).$$

Post-Crash Problem

- Recall: $X_t = (1 - \pi_\tau \ell)\,\tilde{X}_t$, where $\tilde{X}$ is the crash-free setting.
- The solution to the crash-free SDE is given by

$$\tilde{X}_t = x \exp\Bigg(\int_0^t \bigg(r_s + \tilde{\pi}_s \lambda_s - \frac{1}{2}\tilde{\pi}_s^2 \sigma_s^2 + \int_{[0,l^{\max}]} \log(1 - \tilde{\pi}_s l)\vartheta(dl) \bigg)\, ds + \int_0^t \tilde{\pi}_s \sigma_s dW_s + \int_{(0,t]\times[0,l^{\max}]} \log(1 - \tilde{\pi}_s l)\tilde{\nu}(ds, dl) \Bigg),$$

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$$\left. + \int_0^t \tilde{\pi}_s \sigma_s dW_s + \int_{(0,t]\times[0,l^{\max}]} \log(1 - \tilde{\pi}_s l)\tilde{\nu}(ds, dl) \right),$$

- which for $\tau < t$ can be rewritten as

$$\tilde{X}_t = x \exp\left(\int_0^{\tau} \Phi_s(\pi_s) ds + \int_{\tau}^t \Phi_s(\overline{\pi}_s) ds + \int_0^{\tau} \pi_s \sigma_s dW_s + \int_{\tau}^t \overline{\pi}_s \sigma_s dW_s \right.$$
$$\left. + \int_{(0,\tau]\times[0,l^{\max}]} \log(1 - \pi_s l)\tilde{\nu}(ds, dl) + \int_{(\tau,t]\times[0,l^{\max}]} \log(1 - \overline{\pi}_s l)\tilde{\nu}(ds, dl) \right).$$

Post-Crash Problem

- Taking the logarithm, our objective function reads (using boundedness of $\pi, \overline{\pi}$):

$$\mathbb{E}\left[\log X_T^{(\pi,\overline{\pi}),\tau}\right] = \mathbb{E}\left[\log\left(\left(1-\pi_\tau \ell\right)\tilde{X}_T\right)\right]$$

$$= \mathbb{E}\left[\log\left(\left(1-\pi_\tau \ell\right) x \exp\left(\int_0^\tau \Phi_s(\pi_s)ds + \int_\tau^T \Phi_s(\overline{\pi}_s)ds\right)\right)\right]$$

$$= \log x + \mathbb{E}\left[\log\left(1-\pi_\tau \ell\right) + \int_0^\tau \Phi_t(\pi_t)dt\right] + \mathbb{E}\left[\int_\tau^T \Phi_t(\overline{\pi}_t)dt\right].$$

- Thus, **post-crash strategy** as before: $\overline{\pi}_t^* = \pi_t^M = \arg\max_{\overline{\pi}} \Phi_t(\overline{\pi})$
- In the case without Lévy jumps π_t^M is given by $\frac{\lambda_t}{\sigma_t^2}$

Pre-Crash Problem

Rewrite the objective as follows:

$$\mathbb{E}\left[\log\left(1-\pi_\tau \ell\right)+\int_0^\tau \Phi_t(\pi_t)dt\right]+\mathbb{E}\left[\int_\tau^T \Phi_t(\pi_t^M)dt\right]=$$

$$\underbrace{\mathbb{E}\Big[\log\left(1-\pi_\tau \ell\right)+\int_0^\tau \left(\Phi_t(\pi_t)-\Phi_t(\pi_t^M)\right)dt\Big]}_{(A)}+\underbrace{\mathbb{E}\left[\int_0^T \Phi_t(\pi_t^M)dt\right]}_{(B)}$$

Consequences of this representation:

- (B) is independent of τ and π and can therefore be ignored.
- (A) is $\mathcal{F}_\tau$-measurable.
- Our objective is to choose a PRE-CRASH portfolio strategy $\pi \in \mathcal{A}$ as to maximise

$$\sup_\pi \inf_\tau \mathbb{E}\left[\log\left(1-\pi_\tau \ell\right)+\int_0^\tau \left(\Phi_s(\pi_s)-\Phi_s(\pi_s^M)\right)ds\right] \qquad (\mathrm{P}^{SM}_{pre})$$

A BSDE Characterisation of Optimal Strategies

Controller-vs-stopper game approach:

$$\Upsilon_t^\pi := \log(1 - \pi_t \ell) + \int_0^t \left(\Phi_s(\pi_s) - \Phi_s(\pi_s^M)\right) ds \quad \to \text{ martingale!}$$

- Υ_t depends on the path of r_t, λ_t, σ_t! $\Rightarrow$ we cannot solve it through an ODE!

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In such a case, we need a backward stochastic differential equation (BSDE)!

Backward stochastic differential equations

BSDE $\neq$ SDE solved backward in time!

Motivation: conditional expectation

Consider a random variable $\xi \in L^1(\mathcal{F}_T)$ and its conditional expectation,

$$Y_t = \mathbb{E}_t[\xi] := \mathbb{E}[\xi \mid \mathcal{F}_t].$$

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By the martingale representation, we can write $\xi = \mathbb{E}[\xi] + \int_0^T Z_s dW_s$ and get

$$Y_t = \mathbb{E}_t[\xi] = \xi - \int_t^T Z_s dW_s \quad \text{and} \quad Y_T = \xi$$

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So we found two adapted processes (Y, Z) such that, given ξ, $\int_t^T Z_s dW_s$ subtracts the 'right amount of randomness' from ξ to yield an adapted process (which is Y).

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Just like before, but we have an additional function f and want:

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Using the martingale representation again,

$\xi + \int_0^T f(s, Y_s)ds = \mathbb{E}\left[\xi + \int_0^T f(s, Y_s)ds\right] + \int_0^T Z_s dW_s$, we get

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$\xi + \int_0^T f(s, Y_s)ds = \mathbb{E}\left[\xi + \int_0^T f(s, Y_s)ds\right] + \int_0^T Z_s dW_s$, we get

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Backward stochastic differential equations

Next: nonlinear conditional expectation

Just like before, but we have an additional function f and want:

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$\rightarrow$ not explicit anymore in Y! It becomes an equation.

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One may even involve the Z-process:

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Differential notation:

$$dY_t = -f(t, Y_t, Z_t) dt + Z_t dW_t, \quad Y_T = \xi, \quad t \in [0, T].$$

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Back to our problem!

A BSDE Characterisation of Optimal Strategies

Controller-vs-stopper game approach:

$$\Upsilon_t^\pi := \log(1 - \pi_t \ell) + \int_0^t \left(\Phi_s(\pi_s) - \Phi_s(\pi_s^M)\right) ds \quad \rightarrow \text{ martingale!}$$

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Proposition [Utility Crash Exposure BSDE, DMMSt2024+]

Assume that $\mathbb{E}\left[\int_0^T |r_t| dt + \left(\int_0^T |\lambda_t| + |\sigma_t|^2 dt\right)^2\right] < \infty$ (B2), λ, σ $\mathcal{F}^W$-measurable, let ϱ be a stopping time with $0 \le \varrho \le T$, $\pi \in \mathcal{A}$. Then:

1. (i) π is an indifference strategy on $[\varrho, T] \cup \{\infty\}$ and, equivalently,
2. (ii) $\exists\, Z \in \mathbb{L}^2$, such that (Y, Z) is on $[\varrho, T]$ a solution to the BSDE

$$dY_t = \left(\Phi_t\left(\frac{1-e^{-Y_t}}{\ell}\right) - \Phi_t(\pi_t^M)\right) dt + Z_t dW_t, \qquad Y_T = 0,$$

where $\pi = \frac{1-e^{-Y_t}}{\ell}$ and the utility crash exposure Y^π of strategy $\pi \in \mathcal{A}$ is defined by $Y_t^\pi := -\log(1-\pi_t\ell)$.

Existence and Uniqueness

Theorems and corollaries (DMMSt2024+) that allow us to find Y and π:

- Under the assumption $(B\exp)$ that for some $\varepsilon > 0$, $\mathbb{E}\big[\int_0^T |r_t| dt + \int_0^T \exp(\varepsilon(|\lambda_t| + |\sigma_t^2|)) dt\big] < \infty$, there is a unique pair $(Y, Z) \in \mathbb{L}^2 \times \mathbb{L}^2$ which solves the utility crash exposure BSDE. Also, Y is ($\lambda_{[0,t]} \otimes \mathbb{P}$-a.e.) nonnegative and bounded.

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- Under assumption $(B\exp)$ there is a unique indifference strategy π.
- If $\pi \leq \pi^M$, then π is pre-crash optimal
- In particular, this is the case if $\pi^M \equiv \alpha$ is constant.

Markovian Case – PDE-BSDE connection

Market model with $\sigma_t = \overline{\sigma}(z_t)$, $\lambda_t = \overline{\lambda}(z_t)$ where z is a factor process whose evolution is governed by the SDE

$$dz_t = \mu(z_t)dt + \varsigma(z_t)dB_t.$$

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Let π^M be given by $\psi(\lambda, \sigma)$ and let $v \in C^{1,2}$ be a solution to

$$\begin{aligned} 0 =& \partial_t v(t,x) + \mu(x)\partial_x v(t,x) + \frac{\overline{\sigma}^2 x}{2}\partial_{xx} v(t,x) + (\Phi_t(\psi(\overline{\lambda}(x), \overline{\sigma}(x))) - r_t) \\ & - \bar{\lambda}(x)\frac{1 - e^{-(v(t,x)\vee 0)}}{\ell} + \frac{\overline{\sigma}(x)^2}{2}\left(\frac{1 - e^{-(v(t,x)\vee 0)}}{\ell}\right)^2 \\ & - \int_{[0,l^{\max}]} \log\left(1 - \frac{1 - e^{-(v(t,x)\vee 0)}}{\ell} l\right)\vartheta(dl), \quad v(T,x) = 0 \end{aligned}$$

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- Now suppose that $Y_t := v(t, z_t)$ and $Z_t := \varsigma(z_t)\partial_x v(t, z_t)$ are in $\mathbb{L}^2$.
- Then (Y, Z) is the unique $\mathbb{L}^2$-solution to the utility crash exposure BSDE.
- Proof: Just apply Itô's formula to $Y_t := v(t, z_t)$.

Concrete Example: Heston and Bates Model

In Bates' stochastic volatility model, the stock price evolves like

$$dS_t = S_{t-}\left[(\lambda + r)dt + \sqrt{z_t}dW_t - \int_{[0,l^{\max}]} l\nu(dt, dl)\right],$$

and the evolution of z with the corresponding specifications $z = \sigma^2$, $\sigma(x) = \sqrt{x}$ is the Cox-Ingersoll-Ross (CIR) process given by

$$dz_t = \kappa(\theta - z_t)dt + \varsigma\sqrt{z_t}dB_t$$

where B is a second Brownian motion that can be correlated with W.

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Assume an appropriate price of risk

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We have to solve the PDE

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$$+ \frac{x}{2}\left(\frac{1-e^{-(v(t,x)\vee 0)}}{\ell}\right)^2 - \int_{[0,l^{\max}]} \log\left(1 - \frac{1-e^{-(v(t,x)\vee 0)}}{\ell} l\right)\vartheta(dl) = 0, \;\; v(T,x) = 0$$

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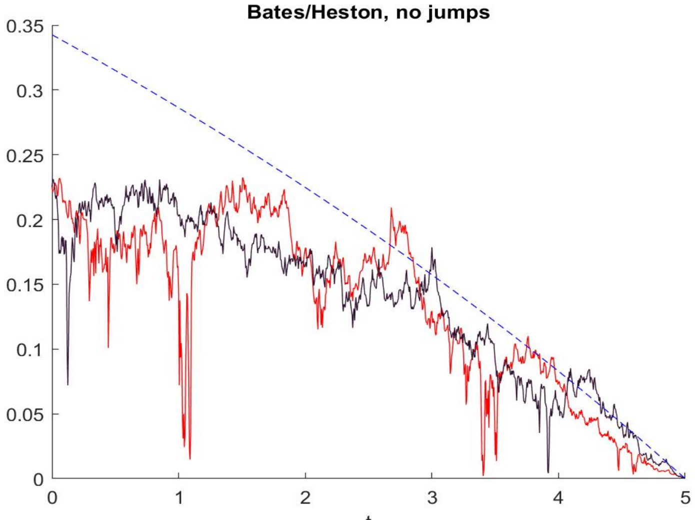
Further, if the Feller condition $\frac{2\kappa\theta}{\varsigma^2} > 1$ is satisfied, then there is $\varepsilon > 0$ such that $\mathbb{E}[\exp(\varepsilon z_t^s(x))] < \infty$ i.e. $(B\exp)$ is satisfied.

Illustration: π_{Bates} (full paths) vs π_{BS} (dashed)



Parameters: $\alpha = 2.5$, $\theta = z_0 = 0.014$, $\kappa = 3.99$, $\varsigma = 0.27$, $\ell = 0.5$, $T = 5$

$\vartheta \equiv 0$, $\lambda_t = \alpha z_t$

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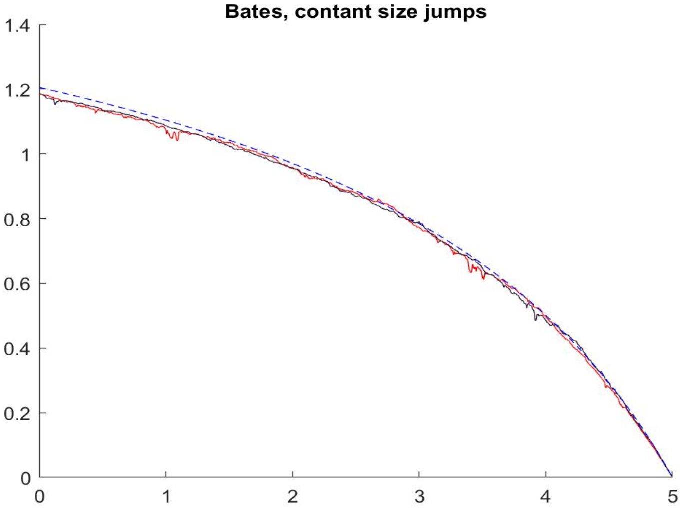

Parameters: $\alpha = 2.5$, $\theta = z_0 = 0.014$, $\kappa = 3.99$, $\varsigma = 0.27$, $\ell = 0.5$, $T = 5$

$q = l^{\max} = 0.2$, $\vartheta = \delta_q$, $\lambda_t = \alpha z_t + \frac{q}{1-\alpha q}$

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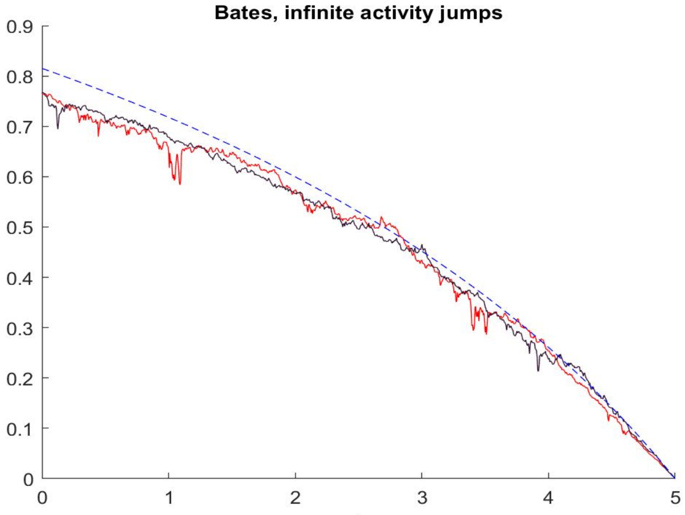

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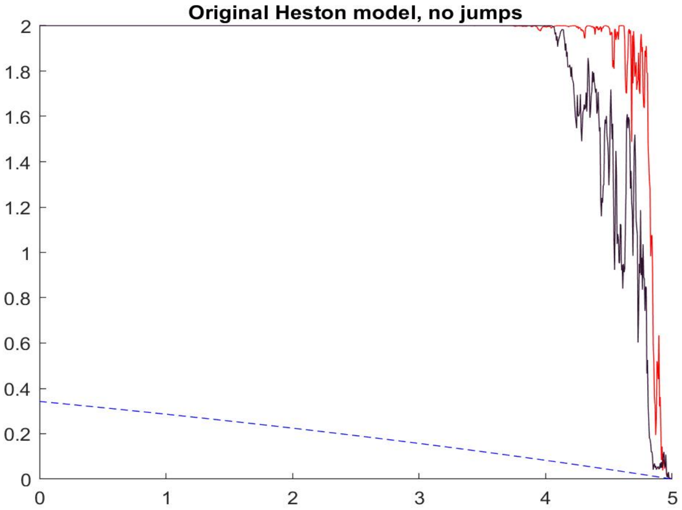
$$I^{\max} = 0.2,\ \vartheta(dl) = \frac{1}{l}dl,\ \lambda_t = \alpha z_t - \frac{\log(1-\alpha I^{\max})}{\alpha}$$

Illustration: π_{Heston} (full paths) vs π_{BS} (dashed)



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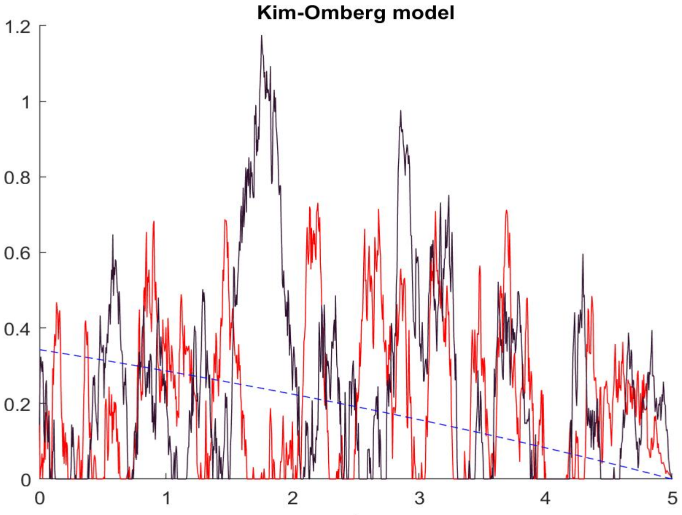
$\vartheta(dl) = 0$, $\lambda_t = \alpha\theta$, $\pi \not\leq \pi^M \neq$const

Illustration: $\pi_{Kim-Omberg}$ (full paths) vs π_{BS} (dashed)



Parameters: $\theta = z_0 = 0.014,\ \kappa = 3.5,\ \varsigma = 0.3,\ \sigma = \sqrt{\theta},\ \ell = 0.5,\ T = 5$

$\vartheta(dl) = 0,\ d\lambda_t = \kappa(\theta - \lambda_t)dt + \varsigma dW_t,$ $\pi \not\leq \pi^M \neq$const

Perspectives

- What happens if λ, σ are fully Lévy-dependent?
- Jump (small crash) sizes governed by a process $g(l)$ instead of constant l.
- What happens if $\pi \nleq \pi^M$.
- Find ways to treat other utility functions such as Power Utility (no additive structure)!

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Thank you for your attention!