A Non-Markov Approach for the Evolution of Contagion in a Network

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Motivation: phenomena of contagion in a network

Features:

- Continuous-time model
- The contagion process evolves in a *random environment* (set of observable, simulable processes)
- The environment also may be affected by the contagion process.

Contagion processs:

$$\text{Entity } i \qquad \underset{\text{survival}}{0} \longrightarrow \underset{\text{default}}{1} \qquad i \in \mathcal{N} \text{ countable set.}$$

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- We propose a model where the contagion processs may "contaminate" its environment.

  A Non-Markov Approach for the Evolution of Contagion in a Network
  D. Coculescu, G. Visentin Working Paper, University of Zürich, 2022.

- Contagion: The transition of one debtor (from state 0 to state 1) impacts the transition probabilities of the surviving debtors.
- Two kinds of contagion:
  - **direct:** a default event directly impacts the intensities surviving debtors in the system; no impact on the environment.
  - **indirect (overspilling)**: a default event is affecting the environment. Indirectly, the intensities surviving debtors in the system are also impacted.

Basic notions and notations

- $(\Omega, \mathcal{G}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ filtered probability space (usual assumptions).
- $\mathbb{F}$ synthesises the relevant information about the environment of the contagion processs.
- We consider n entities with transition times $\tau(k)$, $k \in \mathcal{N} := \{1, ..., n\}$.
- The global information $\mathbb{G}^{\mathcal{N}} = (\mathcal{G}_t^{\mathcal{N}})_{t\geq 0}$ is:

$$\mathcal{G}_t^{\mathcal{N}} = \mathcal{F}_t \bigvee_{k\in\mathcal{N}} \sigma(t \wedge \tau(k)).$$

- The default indicator process $Y_t = (Y_t(1), ..., Y_t(n))$:

$$Y_t(k) = \mathbf{1}_{\{\tau(k)\leq t\}}$$

- The intensities of the transition times $\lambda_t^{\mathcal{N}}(k)$ such that the following are martingales:

$$Y_t(k) - \int_0^{t\wedge\tau(k)} \lambda_s^{\mathcal{N}}(k)ds, \quad t \geq 0$$

Models with interacting intensities

- Conditionally on $\mathcal{F}_\infty$, the default indicator process Y is an inhomogeneous Markov chain, with state space $I = \{0,1\}^n$.
- Contagion is explicitly modelled as being driven by the process Y: the transition rates of Y from state x to state y at time t are of the form:

$$q_t(x,y) = \begin{cases} \mathbf{1}_{\{x(k)=0\}} f_t(x,y) \text{ if } y = x^k \text{ for some } k \in \mathcal{N} \\ 0 \text{ else.} \end{cases}$$

  where:
  - $x^k \in I$ is obtained from $x \in I$ by flipping the k^{th} coordinate.
  - $(f_t(x,y))_{t\geq 0}$ is an $\mathbb{F}$ adaptated process (for fixed $x, y \in I$).
- Therefore, the default intensity of k^{th} obligor is modeled as a function of the default process:

$$\lambda_t(k) = q_t(Y_t, Y_t^k).$$

  - Advantages: Markov chain techniques are available for analysis and simulation, can be used to price complex products as CDOs
  - Disadvantage: Neglects the feedback form contagion in the real economy.

Some related literature

- See: Kusuoka (1999), Davis and Lo (2001), Jarrow and Yu (2001), Bielecki, Rutkowski (2003), Frey and Backhaus (2008,2010), Bielecki, Crépey, Jeanblanc (2007), Rutkowski (2003), Herbertsson (2007) and Herbertsson and Rootzén (2008)

Other existing models are variants of the above framework:

- state 1 is not absorbing: Giesecke and Weber (2004,2006).
- more states: $I = \{1, ..., m\}^n$ credit migration model by Davis, Esparragoza-Rodriguez (2007).
- Frailty models: the filtration $\mathbb{F}$ is (partially) unavailable for pricing (unobserved factors), so that filtering techniques are used: Frey and Schmidt (2009).
- More than one default can possibly occur at a time: Bielecki, Cousin, Crépey, Herbertsson (2011).

See also the survey paper by Bielecki, Crépéy, Herbertsson (2011).

Related work, not intensity based: El Karoui, Jeanblanc, Jiao (2015).

A model with overspilling contagion

2-step construction:

1. We built the model under a measure $\mathbf{P}^0$ that ensures::
   - conditional independence;
   - no contagion.
2. We shape the wished contagion mechanism (direct+indirect) via a suitable change of the probability measure $\mathbf{P}^{\mathcal{N}} \sim \mathbf{P}^0$.

Step 1. The model under P^0

Generalisation of the model by Lando (1998)

- $(\Omega, \mathcal{G}, \mathbb{F}, \mathbf{P}^0)$
- $\Gamma(i)$, $i \in \mathcal{N}$, satisfies:

  $\Gamma_0(i) = 0$ a.s. and $\lim_{t\to\infty} \Gamma_t = +\infty$ a.s.;
  $\mathbb{F}$ adapted, RCLL, increasing process.
- $e(i)$, $i = 1, ..., n$ i.i.d.

  Exp(1), independent of $\mathcal{F}_\infty$.
- We define for $i = 1, ..., n$:

$$\boxed{\tau(i) = \inf\{t \geq 0;\ \Gamma_t(i) \geq e(i)\}.}$$

C. (2010)
Aksamit, Choulli, Jeanblanc (2021)

Step 1. (continued)

Working example:

- For all $i \in \mathcal{N}$, the process $\Gamma(i)$ satisfies:

$$\Gamma_t(i) = \int_0^t \alpha_s(i)ds + \Delta\Gamma_{T(i)}(i)\mathbf{1}_{\{T(i)\leq t\}},$$

$T(i)$ is a totally inaccessible $\mathbb{F}$ stopping time admiting an intensity $(\gamma_t(i))$.

- It follows that the following are $(\mathbb{F}, \mathbf{P}^0)$ *orthogonal* martingales:

$$n_t(i) := \mathbf{1}_{\{T(i)\leq t\}} - \int_0^{t\wedge T(i)} \gamma_s(i)ds.$$

- All processes γ and α are considered bounded.

Azéma supermartingales $Z(k)$ under P^0

Azéma's supermartingale for $\tau(i)$, relative to $\mathbf{P}^0$ and $\mathbb{F}$ is defined as:

$$Z_t(i) := \mathbf{P}^0(\tau(i) > t|\mathcal{F}_t) = e^{-\Gamma_t(i)}.$$

Proposition

$$Z_t(k) = \mathcal{E}_t(\nu(k)) \times \exp\left(-\int_0^t \lambda_s(k)ds\right),$$

where $\nu_t(k) := -\int_0^t g_s(k)dn_s(k)$*;* $g_t(k) := p_t(k)e^{\Gamma_{t-}(k)}\mathbf{1}_{\{T(k)\geq t\}}$ *and with* $\lambda(k)$*, the intensity of* $\tau(k)$*, being:*

$$\lambda_t(k) := \alpha_t(k) + \underbrace{g_t(k)\gamma_s(k)}_{:=\beta_t(k)}.$$

Survival probabilities

The probability $\mathbf{P}^0$ of survival in group $\mathcal{C} \subset \mathcal{N}$ and at time t is:

$$
\begin{aligned}
\mathbf{P}^0(\tau(k) > t,\ \forall k \in \mathcal{C}) &= \mathbf{E}^0\left[\prod_{k\in\mathcal{C}} Z_t(k)\right] \\
&= \mathbf{E}^0\left[\exp\left(-\sum_{k\in\mathcal{C}}\int_0^t \lambda_s(k)ds\right)\prod_{k\in\mathcal{C}}\mathcal{E}_t(\nu(k))\right] \qquad (1)\\
&= \bar{\mathbf{E}}_{\mathcal{C}}\left[e^{-\int_0^t \lambda_s(\mathcal{C})ds}\right] = \bar{\mathbf{E}}_{\mathcal{C}}\left[\ell_t\right], \qquad (2)
\end{aligned}
$$

ℓ satsfies: $d\ell_t = -\ell_t\lambda_t(\mathcal{C})dt$; and $\bar{\mathbf{E}}_{\mathcal{C}}$ is the expectation under $\bar{\mathbf{P}}_{\mathcal{C}}$ (see below).

Definition (The default-adjusted probability measure)

For $\mathcal{C} \subset \mathcal{N}$, we define the following auxiliary probability measure $\bar{\mathbf{P}}_{\mathcal{C}}$:

$$
\left.\frac{d\bar{\mathbf{P}}_{\mathcal{C}}}{d\mathbf{P}^0}\right|_{\mathcal{G}_t^{\mathcal{N}}} = \prod_{k\in\mathcal{C}}\mathcal{E}_t(\nu(k)), \quad t \geq 0.
$$

The structure of the information

- With a subset $\mathcal{C} \subset \mathcal{N}$, we associate a filtration $\mathbb{G}^{\mathcal{C}}$:

$$\mathcal{G}_t^{\mathcal{C}} = \mathcal{F}_t \vee_{k \in \mathcal{C}} \sigma(t \wedge \tau(k)),$$

  i.e., the progressively enlarged filtration that makes all $\tau(k)$, $k \in \mathcal{C}$ stopping times.
- **Notation.** For two filtrations $\mathbb{F} \subset \mathbb{G}$ and a probability $\mathbf{P}$, we write $\mathbb{F} \overset{\mathbf{P}}{\hookrightarrow} \mathbb{G}$ whan $\mathbb{F}$ martingales remain $\mathbb{G}$ martingales under the probability $\mathbf{P}$.
- **Remark.** For any $\mathcal{C} \subset \mathcal{N}$ and $k \in \mathcal{C}$ we have the following property:

$$\mathbb{F} \overset{\mathbf{P^0}}{\hookrightarrow} \mathbb{G}^{\mathcal{C}} \overset{\mathbf{P^0}}{\hookrightarrow} \mathbb{G}^{\mathcal{N}}.$$

$$\mathbb{F} \overset{\bar{\mathbf{P}}_{\mathcal{C}}}{\hookrightarrow} \mathbb{G}^{\mathcal{C}} \overset{\bar{\mathbf{P}}_{\mathcal{C}}}{\hookrightarrow} \mathbb{G}^{\mathcal{N}}.$$

The decomposition of the default times $\tau(k)$

Proposition

Consider $k \in \mathcal{C}$. We define the $\mathbb{G}^{\mathcal{C}}$ stopping times $\tau^A(k)$ and $\tau^B(k)$:

$$\tau^A(k) := \tau(k)\mathbf{1}_{\{\tau(k)\neq T(k)\}} + \infty\mathbf{1}_{\{\tau(k)=T(k)\}}$$
$$\tau^B(k) := \tau(k)\mathbf{1}_{\{\tau(k)=T(k)\}} + \infty\mathbf{1}_{\{\tau(k)\neq T(k)\}},$$

so that:

$$\tau(k) = \tau^A(k) \wedge \tau^B(k).$$

Then:

- *the $(\mathbb{G}^{\mathcal{C}}, \mathbf{P}^0)$ intensity of $\tau^A(k)$ is $\alpha_t(k)\mathbf{1}_{\{\tau(k)>t\}}$;*
- *the $(\mathbb{G}^{\mathcal{C}}, \mathbf{P}^0)$ intensity of $\tau^B(k)$ is $\beta_t(k)\mathbf{1}_{\{\tau(k)>t\}}$.*

Example 1.

Suppose that:

$$\tau(k) = \begin{cases} T^0(k), & \mathbf{P}(\tau(k) = T^0(k)|\mathcal{F}_t) = 1 - p \\ T(k), & \mathbf{P}(\tau(k) = T(k)|\mathcal{F}_t) = p \end{cases}$$

- $T^0(k) \sim exp(\alpha)$ firm-specific factor, independent from $\mathbb{F}$;
- $T(k) \sim exp(\gamma)$ macro factor, $\mathbb{F}$ stopping time.

Then:

- $(\mathbb{F}, \mathbf{P}^0)$-survival process of entity k is

$$\mathbf{P}^0(\tau(i) > t|\mathcal{F}_t) = (1-p)e^{-\alpha t} + p\mathbf{1}_{\{\mathbf{T(k)>t}\}}$$

- the $(\mathbb{G}^k, \mathbf{P}^0)$-intensity of $\tau(k)$ is:

$$\lambda_t(k) = \alpha + \underbrace{1_{\{T(k)\geq t\}}\frac{p(\gamma-\alpha)}{p+(1-p)e^{-\alpha t}}}_{=\beta_t(k)}$$

The main martingales

We summarise the important $(\mathbb{G}^{\mathcal{C}}, \mathbf{P}^0)$ martingales:

$$m_t(k) = \mathbf{1}_{\{\tau^A(k) \le t\}} - \int_0^{t \wedge \tau(k)} \alpha_s(k) ds, \quad t \ge 0$$

$$n_t(k) = \mathbf{1}_{\{T(k) \le t\}} - \int_0^{t \wedge T(i)} \gamma_s(k) ds, \quad t \ge 0.$$

Creating a group $\mathcal{C} \subset \mathcal{N}$ of contagious debtors

We consider filtered probability spaces

$$(\Omega, \mathcal{G}, \mathbb{G}^{\mathcal{N}}, \mathbf{P}^{\mathcal{C}}), \mathcal{C} \subset \mathcal{N}$$

corresponding to different groups $\mathcal{C}$ of contagious debtors.

Aim:

We want that for $i \in \mathcal{N}$ $\tau(i)$ has a $(\mathbb{G}^{\mathcal{N}}, \mathbf{P}^{\mathcal{C}})$-intensity:

$$\lambda_t^{\mathcal{C}}(i) = \lambda_t(i) + \sum_{j \in \mathcal{C}} \xi_t^{X(j)}(i,j) \mathbf{1}_{\{\tau(j) < t\}} \quad \text{for } i \in \mathcal{N}.$$

with $X(j) = A\mathbf{1}_{\{\tau(j)=\tau^A(j)\}} + B\mathbf{1}_{\{\tau(j)=\tau^B(j)\}}$, which is a $\mathcal{G}^{\mathcal{N}}_{\tau(j)}$ measurable random variable; $X(j) = A$ if the default j is producing a direct contagion, while $X(j) = B$ will indicate that we have indirect contagion.

We introduce the impact matrices:

- $(\phi_t^A(i,j))_{(i,j)\in\mathcal{N}^2}$ **direct contagious impact** of debtor j on debtor i.
- $(\phi_t^B(i,j))_{(i,j)\in\mathcal{N}^2}$ **indirect contagious impact**.

with components being positive, bounded and $\mathbb{F}$-predictable processes.

Creating a group $\mathcal{C} \subset \mathcal{N}$ of contagious debtors

Proposition

Let $\mathcal{C} \subset \mathcal{N}$. Let us introduce for all $i \in \mathcal{C}$ the predictable processes:

$$A_t^{\mathcal{C}}(i) := \frac{1}{\alpha_t(i)} \sum_{j \in \mathcal{C}} \phi_t^A(i,j) \mathbf{1}_{\{\tau^A(j) < t\}}, \quad t \geq 0 \tag{3}$$

$$B_t^{\mathcal{C}}(i) := \frac{1}{\gamma_t(i)} \sum_{j \in \mathcal{C}} \phi_t^B(i,j) \mathbf{1}_{\{\tau^B(j) < t\}}, \quad t \geq 0 \tag{4}$$

and define the family of probability measures $(\mathbf{P}^{\mathcal{C}}), \mathcal{C} \subset \mathcal{N}$:

$$\left.\frac{d\mathbf{P}^{\mathcal{C}}}{d\mathbf{P}^0}\right|_{\mathcal{G}_t^{\mathcal{N}}} = D_t^{\mathcal{C}} := \prod_{i \in \mathcal{N}} \mathcal{E}_t \left(\int_0^t A_s^{\mathcal{C}}(i) dm_s(i) \right) \prod_{i \in \mathcal{N}} \mathcal{E}_t \left(\int_0^t B_s^{\mathcal{C}}(i) dn_s(i) \right).$$

Then the default time $\tau(i)$, $i \in \mathcal{N}$ has the $(\mathbb{G}^{\mathcal{N}}, \mathbf{P}^{\mathcal{C}})$ intensity $\lambda^{\mathcal{C}}(i)$ given by

$$\lambda_t^{\mathcal{C}}(i) = \lambda_t(i) + \sum_{j \in \mathcal{C}} \left(\phi_t^A(i,j) \mathbf{1}_{\{\tau^A(j) < t\}} + g_t(i) \phi_t^B(i,j) \mathbf{1}_{\{\tau^B(j) < t\}} \right)$$

Remark

Under $\mathbf{P}^{\mathcal{C}}$, some defaults may modify the evolution of the environment: the $(\mathcal{G}^{\mathcal{N}}, \mathbf{P}^{\mathcal{C}})$-intensity of a stopping time $T(i), i \in \mathcal{N}$ is

$$\gamma(i)[1 + B_t^{\mathcal{C}}(i)],$$

i.e., has upward jumps at the default times $j \in \mathcal{C}$ that satisfy $\tau(j) = \tau^B(j)$. Or, $T(i)_{i \in \mathcal{N}}$ are $\mathbb{F}$-stopping times hence they are elements of the environment of the contagion processs.

Survival probabilities

Assumptions:

- Time 0, all debtors are in state 0 and contagious.
- Consequenly, we work under $(\Omega, \mathcal{F}, \mathbb{G}^{\mathcal{N}}, \mathbf{P}^{\mathcal{N}})$.

Notations:

$$\lambda_t(\mathcal{C}) := \sum_{k \in \mathcal{C}} \lambda_t(k); \quad \Phi_t^A(\mathcal{C}, j) := \sum_{k \in \mathcal{C}} \phi_t^A(k, j); \quad \text{etc.}$$

In the next theorem, $\mathcal{C}, \mathcal{D} \in \mathcal{N}$ are fixed, with $\mathcal{C} \cap \mathcal{D} = \emptyset$ and $\mathcal{S} := \mathcal{N} - \mathcal{C}$.

Theorem (Part 1)

$$\mathbf{P}^{\mathcal{N}}(\tau(i) > t, \forall i \in \mathcal{C}\ ;\ \tau^B(j) \leq t, \forall j \in \mathcal{D}) = \bar{\mathbf{E}}_{\mathcal{C}}[\ell_t^{\mathcal{S}|\mathcal{D}} \prod_{j \in \mathcal{D}} p_t(j) \mathbf{1}_{\{T(j) \leq t\}}],$$

where $\ell^{\mathcal{S}|\mathcal{D}}$ *satisfies:*

$$\begin{aligned} d\ell_t^{\mathcal{S}|\mathcal{D}} &= -\ell_{t^-}^{\mathcal{S}|\mathcal{D}} \lambda_t(\mathcal{C}) dt \\ &- \sum_{j \in \mathcal{S}-\mathcal{D}} \left(\ell_{t^-}^{\mathcal{S}|\mathcal{D}} - \ell_{t^-}^{\mathcal{S}-j|\mathcal{D}} - \ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \mathbf{1}_{\{T(j)<t\}} \right) \psi^A(\mathcal{C} \cup \mathcal{D}, j) dt \\ &+ \sum_{j \in \mathcal{S}} \mathbf{1}_{\{T(j)<t\}} \left(\mathbf{1}_{\{j \in \mathcal{D}\}} \ell_{t^-}^{\mathcal{S}|\mathcal{D}} + \mathbf{1}_{\{j \in \mathcal{S}-\mathcal{D}\}} \ell_{t^-}^{\mathcal{S}|\mathcal{D}\cup j} p_t(j) \right) \sum_{k \in \mathcal{N}} \frac{\phi_t^B(k,j)}{\gamma_t(k)} dn_t(k). \end{aligned} \tag{5}$$

$\ell_0^{\mathcal{S}|\mathcal{D}} = 1$. *Above, we have denoted:*

$$\psi^A(k,j) := \begin{cases} \phi^A(k,j) & k \in \mathcal{S} - \mathcal{D} \\ \phi^A(k,j) \mathbf{1}_{\{T(k)>t\}} & k \in \mathcal{D}. \end{cases}$$

Theorem (Part 2)

In particular, denoting $\ell^{\mathcal{S}} := \ell^{\mathcal{S}|\emptyset}$*, the survival probability in group* $\mathcal{C}$ *satisfies:*

$$\mathbf{P}^{\mathcal{N}}(\tau(k) > t,\ \forall k \in \mathcal{C}) = \bar{\mathbf{E}}_{\mathcal{C}}\left[\ell^{\mathcal{S}}_t\right],$$

with:

$$\begin{aligned} d\ell^{\mathcal{S}}_t =& -\left\{\ell^{\mathcal{S}}_{t-}\lambda_t(\mathcal{C}) + \sum_{j\in\mathcal{S}}\left(\ell^{\mathcal{S}}_{t-} - \ell^{\mathcal{S}-j}_{t-} - \ell^{\mathcal{S}|j}_{t-}p_t(j)\mathbf{1}_{\{T(j)<t\}}\right)\phi^{A}(\mathcal{C},j)\right\}dt \\ &+ \sum_{j\in\mathcal{S}}\mathbf{1}_{\{T(j)<t\}}\ell^{\mathcal{S}|j}_{t-}p_t(j)\sum_{k\in\mathcal{N}}\frac{\phi^{B}_t(k,j)}{\gamma_t(k)}dn_t(k) \qquad (6) \\ \ell^{\mathcal{S}}_0 =& 1. \end{aligned}$$

Recursive procedure for computing $\ell^{\mathcal{S}^*}$, $\mathcal{S}^* \subset \mathcal{N}$

- If **card**$(\mathcal{S}^*) = s$, are necessary iterations $k = 0, 1, \cdots, s$ of the type:

  k. *For any $\mathcal{S} \subset \mathcal{S}^*$ with* **card**$(\mathcal{S}) = k$ *and for any $\mathcal{D} \subset \mathcal{S}$, we compute $\ell^{\mathcal{S}|\mathcal{D}}$, in the decreasing order of the cardinality of $\mathcal{D}$.*

- There are $\binom{n}{k}$ subsets of $\mathcal{S}^*$ having exactly k elements; each such subset having 2^k different subsets.
- Therefore at the k^{th} iteration, we need to solve $\binom{n}{k}2^k$ equations of the type (5).
- For solving these equations, the quantities obtained at step $k-1$ are needed.
- Overall

$$\sum_{k=0}^{s} \binom{s}{k} 2^k = 3^s$$

equations of the type (5) need to be solved.

Particular cases:

1. If $\phi^A \equiv 0$ and $\phi^B \equiv 0$ (no contagion), then $\mathbf{P}^{\mathcal{C}} = \mathbf{P}^0$ and:

$$d\ell_t^{\emptyset} = -\ell_t^{\emptyset}\lambda_t(\mathcal{C})dt$$

2. If $p(i) \equiv 0$ for all $i \in \mathcal{N}$ (i.e., there is no impact of the contagion processs on its environment), then all $\tau(i), i \in \mathcal{N}$ avoid the $\mathbb{F}$ stopping times. We recover in this way a Markovian framework where the transition rate at time t from state $x \in \{0,1\}^n$ to state $y \in \{0,1\}^n$ is:

$$q_t(x,y) = \begin{cases} \lambda_t(k) + \sum_{j\in\mathcal{N}} \phi_t^A(k,j)x(j) \text{ if for some } k \in \mathcal{N} : y = x^k \text{ and } x(k) = \\ 0 \text{ else,} \end{cases}$$

where, as in the previous section, $x^k \in \{0,1\}^n$ is obtained from $x \in I$ by flipping the k^{th} coordinate, $x(k)$.
We observe that $\bar{\mathbf{P}}_{\mathcal{C}} = \mathbf{P}^0$ and (6) becomes:

$$d\ell_t^{\mathcal{S}} = -\,\ell_t^{\mathcal{S}}\left\{\lambda_t(\mathcal{C}) + \phi_t^A(\mathcal{C},\mathcal{S})\right\}dt + \sum_{j\in\mathcal{S}} \ell_t^{\mathcal{S}-j}\phi_t^A(\mathcal{C},j)dt. \tag{7}$$

Particular cases:

3. If $\phi^A = 0$ and $\phi^B \neq 0$ (i.e., there is only indirect contagion), then:

$$d\ell_t^{\mathcal{S}} = -\,\ell_{t^-}^{\mathcal{S}} \lambda_t(\mathcal{C})dt + \sum_{j \in \mathcal{S}} \mathbf{1}_{\{T(j)<t\}} \ell_{t^-}^{S|j} p_t(j) \sum_{k \in \mathcal{N}} \left(\frac{\phi_t^B(k,j)}{\gamma_t(k)} \right) dn_t(k).$$

Example 2. Numerical application

- $\mathcal{N} = \{1, ..., 5\}$, homogeneous entities
- $T(k) \sim Exp(\gamma)$, $\gamma = 0.2$
- $\alpha_t(1) = ... = \alpha_t(5) = \Psi_t$ that follows

$$d\Psi_t = a(b - \Psi_t)dt + \sigma\sqrt{\Psi_t}dW_t$$

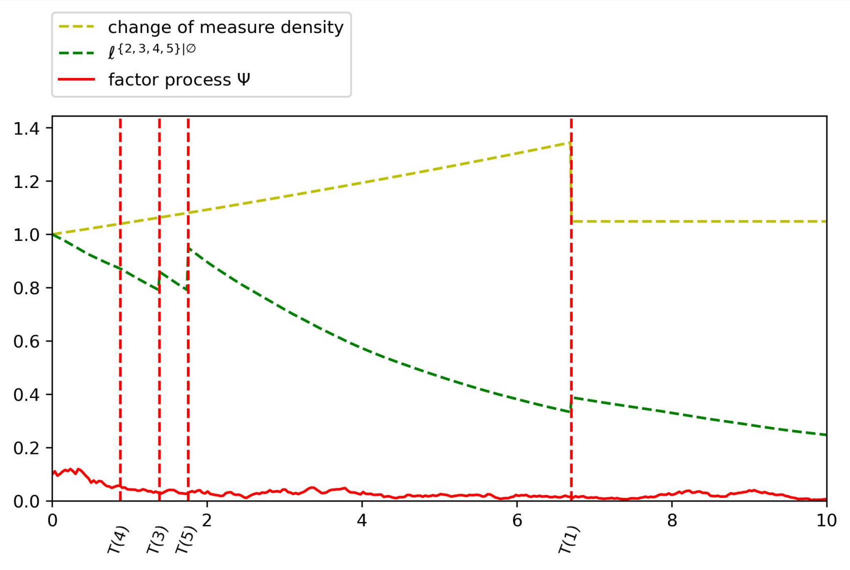
- $\eta_t(1) = ...\eta_t(5) = \eta = 0.25$, $\forall t$
- The hazard process of the default time $\tau(k)$ is therefore

$$\Gamma_t(k) = \int_0^t \Psi_s ds + \eta \mathbf{1}_{\mathbf{T(k) \leq t}}$$

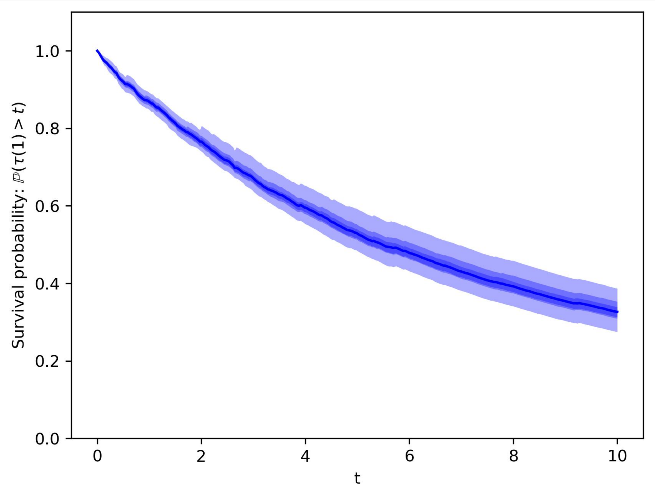
- Impact matrices:

$$\Phi_t^A(i,j) = 0.2 \quad \Phi_t^B(i,j) = 0.2$$

Simulation results

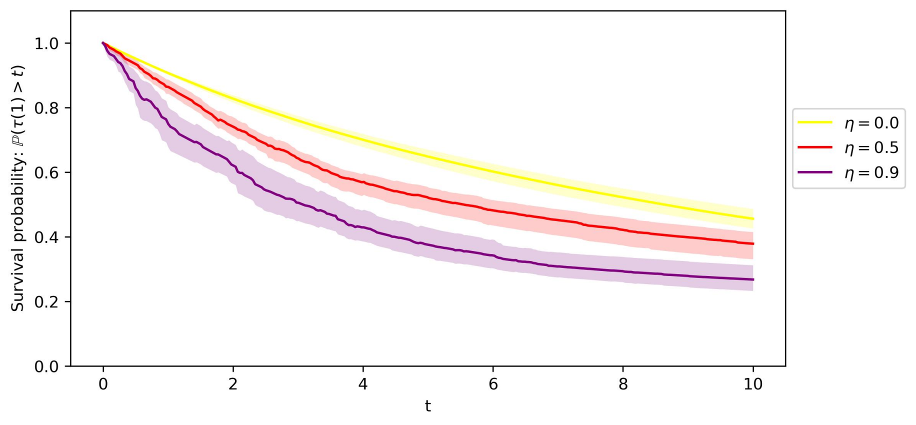


Simulation results



Solid blue: Monte Carlo mean estimator, the blue shaded areas correspond to different levels of confidence for the mean estimator: 50% , 75% , 99%.

Simulation results



Thank you for your attention!