Kendall's tau matrices and attainability of concordance signatures

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When only partial information is available

- ✓ In a risk analysis, we often need to generate scenarios from a model with some prescribed correlation.
- ✓ To obtain this correlation, we may have limited or non-existent data and may need to incorporate expert knowledge.
- ✓ To fix ideas, suppose we wish to take the dependence between the cryptocurrencies Bitcoin, Ethereum, Litecoin and Ripple into account.

  We have joint observations of Bitcoin, Ethereum and Litecoin and an expert opinion on the correlation between Bitcoin and Ripple.

Cast in mathematical terms

Consider a random vector $(X_1, \dots, X_d)$ and a measure of association ϱ.

Let P_ϱ be the matrix with entries $\varrho(X_i, X_j)$, $i, j \in \{1, \dots, d\}$.

- We have knowledge of some, but not all entries of P_ϱ. What is the range of entries that are missing?
- Given a matrix P, can we find a distribution with $P_\varrho = P$?
- What if we have knowledge of some higher-order associations?

Here, we take ϱ to be Kendall's tau.

Matrices of Spearman's rho, Blomquist's beta, Pearson's correlation, and tail dependence coefficients have been investigated by Devroye and Letac (2015), Embrechts et al. (2016), Huber and Marić (2015, 2019), Hofert and Koike (2019), Wang et al. (2019).

Kendall's tau

Kendall's tau is a widely-used measure of correlation between X_1 and X_2:

$$\tau(X_1, X_2) = \Pr\{(X_1 - X_1^*)(X_2 - X_2^*) > 0\} - \Pr\{(X_1 - X_1^*)(X_2 - X_2^*) < 0\}\ ,$$

where (X_1^*, X_2^*) and (X_1, X_2) are independent and identically distributed.

Maurice Kendall (1907–1983)

Basic observation

One can see that if X_1 and X_2 have continuous distributions F_1, F_2, then

$$\tau(X_1, X_2) = 2\underbrace{\Pr\{(X_1 - X_1^*)(X_2 - X_2^*) > 0\}}_{\kappa_{\{1,2\}}} - 1.$$

The concordance probability $\kappa_{\{1,2\}}$ equals

$$\begin{aligned} \kappa_{\{1,2\}} &= 2\Pr(X_1 < X_1^*, X_2 < X_2^*) \\ &= 2\Pr\{F_1(X_1) < F_1(X_1^*), F_2(X_2) < F_2(X_2^*)\} \\ &= 2\Pr(U_1 < U_1^*, U_2 < U_2^*), \end{aligned}$$

where the distribution of

$$(U_1, U_2) = (F_1(X_1), F_2(X_2))$$

is the copula C of (X_1, X_2). Hence, $\tau(X_1, X_2)$ depends only on C.

First question to be asked today

Consider a continuous random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$. The $d \times d$ Kendall's rank correlation matrix is given by

$$P_\tau = \big(\tau(X_i, X_j)\big)_{i,j=1}^{d}.$$

What properties characterize P_τ?

P_τ is symmetric, its diagonal entries equal 1 and off-diagonal entries are elements of $[-1, 1]$.

These properties are necessary but not sufficient.

Concordance signature

Consider again a continuous random vector $\boldsymbol{X} = (X_1, \dots, X_d)$.

For each $I \subseteq \mathcal{D} = \{1, \dots, d\}$, define the concordance probability κ_I:

$$\kappa_I = \kappa(\boldsymbol{X}_I) = 2 \Pr(\boldsymbol{X}_I < \boldsymbol{X}_I^*) = \Pr(\{\boldsymbol{X}_I < \boldsymbol{X}_I^*\} \cup \{\boldsymbol{X}_I^* < \boldsymbol{X}_I\}) \ ,$$

where $\boldsymbol{X}^*$ is an independent copy of $\boldsymbol{X}$.

These probabilities form the concordance signature of $\boldsymbol{X}$

$$\boldsymbol{\kappa}_{\boldsymbol{X}} = (\kappa_I \ : \ I \in \mathcal{P}(\mathcal{D})),$$

where $\mathcal{P}(\mathcal{D})$ is the power set of $\mathcal{D}$, $\kappa_{\{i\}} = 1$ and $\kappa_\emptyset = 1$.

What do we know about concordance signatures?

- ✓ The concordance signature $\boldsymbol{\kappa}_{\boldsymbol{X}}$ of a random vector with continuous margins $F_1, \ldots, F_d$ depends only on the copula of $\boldsymbol{X}$, i.e., the distribution function of

$$\boldsymbol{U} = (F_1(X_1), \ldots, F_d(X_d)).$$

- ✓ For any set $I = \{i, j\}$ with two elements $i \neq j$, we have that

$$\tau(X_i, X_j) = 2\kappa_{\boldsymbol{I}} - 1.$$

- ✓ For any subset I with $|I| > 2$,

$$(2^{|I|-1} - 1)\tau(\boldsymbol{X}_I) = 2^{|I|-1}\kappa_{\boldsymbol{I}} - 1 \ .$$

Some sets are not needed

From the inclusion-exclusion principle, we have that when $|I|$ is odd,

$$\begin{aligned}\kappa_I = 2\Pr(\boldsymbol{X}_I < \boldsymbol{X}_I^*) &= 1 + \sum_{A\subset I, 1\le|A|<|I|} (-1)^{|A|}\Pr(\boldsymbol{X}_A < \boldsymbol{X}_A^*) \\ &= 1 - |I|/2 + \sum_{A\subset I, 1\le|A|<|I|} (-1)^{|A|}\kappa_A/2\,,\end{aligned}$$

as shown in Genest et al. (2011).

The even concordance signature of C is

$$\boldsymbol{\kappa}^{\mathcal{E}} = (\kappa_I : I \in \mathcal{E}(\mathcal{D})),$$

where $\mathcal{E}(\mathcal{D}) \subset \mathcal{P}(\mathcal{D})$ consists of the subsets of even cardinality, including the empty set. Recall that $\kappa_\emptyset = 1$.

An easy example

Consider a Gaussian random vector $\boldsymbol{X}$. Then

$$\kappa_{\boldsymbol{I}} = 2\Pr(\boldsymbol{X}_{\boldsymbol{I}} < \boldsymbol{X}^*_{\boldsymbol{I}}) = 2\Pr(\boldsymbol{X}_{\boldsymbol{I}} < 0).$$

Suppose that $d = 4$ and the correlation matrix P of $\boldsymbol{X}$ is an equicorrelation matrix with $P_{ij} = \varrho = 0.7$ for $i \neq j$. Then

$$\boldsymbol{\kappa}^{\mathcal{E}} \approx (1, 0.747, 0.747, 0.747, 0.747, 0.747, 0.747, 0.541)\ .$$

For any subset I with $|I| = 3$, such as $I = \{1, 2, 3\}$, we have

$$\kappa_{\boldsymbol{I}} \approx 0.620 = 1 - 3/2 + 3 \times 0.747/2.$$

Second question to be asked today

What properties characterize the even concordance signature $\boldsymbol{\kappa}^{\mathcal{E}}$?

We know that $\boldsymbol{\kappa}^{\mathcal{E}}$ is a vector of length

$$\sum_{j=0}^{\lfloor d/2 \rfloor} \binom{d}{2j} = 2^{d-1}$$

whose first element is 1 and other elements lie in $[0, 1]$. What else?

To answer these questions, we first need to take a detour...

Extremal copulas

Consider an index set $J \subseteq \mathcal{D} = \{1, \ldots, d\}$ and a random vector $\boldsymbol{U}$ with

$$U_j = \begin{cases} U & \text{if } j \in J, \\ 1 - U & \text{if } j \notin J, \end{cases}$$

where U is uniform on $[0, 1]$.

The vector $\boldsymbol{U}$ has uniform margins and spreads its mass uniformly along a main diagonal of the unit hypercube $[0, 1]^d$. Its cdf is the extremal copula

$$C(u_1, \ldots, u_d) = \left(\min_{j \in J} u_j + \min_{j \in J^\complement} u_j - 1 \right)^+ .$$

What's in a name?

Consider an extremal copula C with index set $J \subseteq \in \{1, \dots, d\}$.

The Kendall rank correlation matrix of C is such that

$$(P_\tau)_{ij} = \begin{cases} 1 & \text{if } i, j \in J \text{ or } i, j \in J^{\complement}, \\ -1 & \text{otherwise.} \end{cases}$$

In other words, P_τ is an extremal correlation matrix. We can also write

$$P_\tau = (2\boldsymbol{s} - \mathbf{1})(2\boldsymbol{s} - \mathbf{1})^\top,$$

where $\boldsymbol{s}$ is a vector of length d with $s_k = 0$ if $k \in J$ and $s_k = 1$ otherwise.

In fact, P_τ is also the Spearman and Pearson correlation matrix of C.

Signature of an extremal copula

For $I \subseteq \mathcal{D}$, let

$$a_I = \begin{cases} 1 & \text{if } I \subseteq J \text{ or } I \subseteq J^{\complement}, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $a_{\{i\}} = 1$; we also set $a_\emptyset = 1$ by convention.

Take $\boldsymbol{U} \sim C$ and let $\boldsymbol{U}_I$ be the subvector with indices in I. The quantity a_I is an indicator which states whether the components of $\boldsymbol{U}_I$ are comonotonic variables or not.

The signature of an extremal copula with index set J is

$$\boldsymbol{a}_J = (a_I, I \in \mathcal{P}(D)).$$

Enumerating extremal copulas

Each of the 2^{d-1} main diagonals of $[0,1]^d$ corresponds to one extremal copula:

- ✓ The k-th main diagonal of the hypercube is a line joining its vertices $\boldsymbol{s}_k$ and $\mathbf{1} - \boldsymbol{s}_k$, where $\boldsymbol{s}_k = (s_{k,1}, \ldots, s_{k,d})$ of $k-1$ when represented as a d-digit binary number.

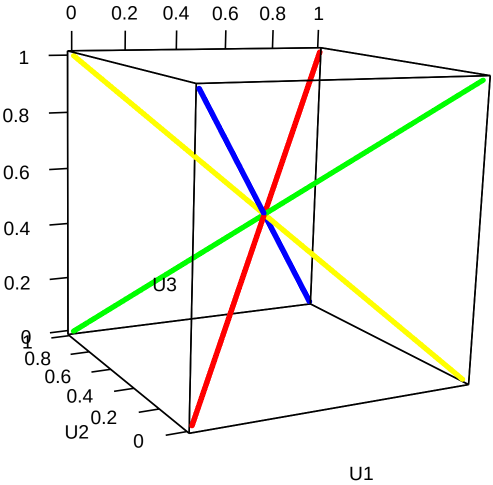
  For example, when $d = 3$ we have

$$\boldsymbol{s}_1 = (0,0,0), \quad \boldsymbol{s}_2 = (0,0,1), \quad \boldsymbol{s}_3 = (0,1,0), \quad \boldsymbol{s}_4 = (0,1,1).$$

- ✓ The k-th extremal copula $C^{(k)}$ spreads its mass along the k-th main diagonal. Its index set J_k corresponds to the zeros in $\boldsymbol{s}_k$, viz.

$$j \in J_k \quad \Leftrightarrow \quad s_{k,j} = 0.$$

Illustration in 3D



Samples of size 2000 from extremal copulas $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ and $C^{(4)}$ in $d = 3$.

Extremal mixtures

An extremal mixture copula has the form

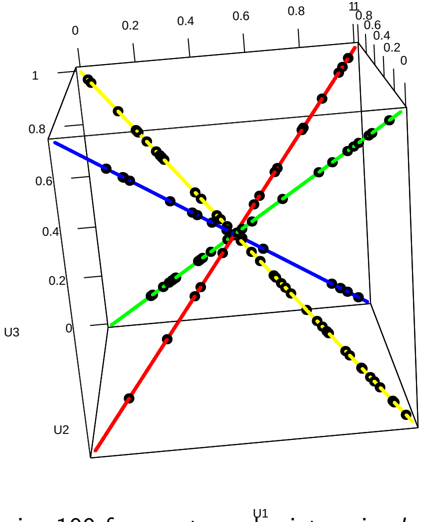
$$C = \sum_{k=1}^{2^{d-1}} w_k C^{(k)} ,$$

where $C^{(k)}$, $k \in \{1, \ldots, 2^{d-1}\}$ are extremal copulas. The weights satisfy

a) $w_1 \geq 0, \ldots, w_{2^{d-1}} \geq 0$;

b) $w_1 + \cdots + w_{2^{d-1}} = 1$.

Illustration in 3D



A sample of size 100 from extremal mixture in $d = 3$ with weights $\boldsymbol{w} = (0.1, 0.2, 0.3, 0.4)$.

Signatures of extremal mixtures

Theorem 1: Signatures of extremal mixtures

For an extremal mixture copula $C = \sum_{k=1}^{2^{d-1}} w_k C^{(k)}$

$$\kappa(C_I) = \sum_{k=1}^{2^{d-1}} w_k \kappa\left(C_I^{(k)}\right)$$

for any $I \subseteq \mathcal{D}$.

This results means that for any extremal mixture C,

$$\boldsymbol{\kappa}_C = \sum_{k=1}^{2^{d-1}} w_k \boldsymbol{\kappa}_{C^{(k)}} = \sum_{k=1}^{2^{d-1}} w_k \boldsymbol{a}_{J_k}.$$

A stunning result

Theorem 2: Signatures of arbitrary copulas

Let C be a d-dimensional copula and $\boldsymbol{\kappa}_C = (\kappa_I, I \in \mathcal{P}(\mathcal{D}))$ its concordance signature.

Then there exists a unique extremal mixture copula with the same concordance signature.

That is, the set of all attainable concordance signatures of d-dimensional random vectors with continuous margins is the convex hull

$$\Big\{ \sum_{k=1}^{2^{d-1}} w_k \boldsymbol{a}_{J_k} \ : \ \boldsymbol{w} \geq \mathbf{0}, \sum_{k=1}^{2^{d-1}} w_k = 1 \Big\}.$$

The reasons behind

- ✓ An extremal mixture copula C^{E} with weights $\boldsymbol{w}$ is the copula of

$$U\boldsymbol{Y} + (1-U)(\mathbf{1}-\boldsymbol{Y}),$$

where U is uniform and independent of the Bernoulli random vector $\boldsymbol{Y}$ with the property that for all $k \in \{1, \ldots, 2^{d-1}\}$,

$$\Pr(\boldsymbol{Y} = \boldsymbol{s}_k) = \Pr(\boldsymbol{Y} = \mathbf{1} - \boldsymbol{s}_k) = w_k/2.$$

- ✓ Such a $\boldsymbol{Y}$ is radially symmetric, or palindromic, viz. $\boldsymbol{Y} \stackrel{\mathrm{d}}{=} \mathbf{1} - \boldsymbol{Y}$.
- ✓ The correspondence between palindromic Bernoulli random vectors and extremal copulas is one-to-one.

The reasons behind (cont'd)

- ✓ Take $\boldsymbol{U}$ and $\boldsymbol{U}^*$ independent and distributed as C. Then

$$\kappa_I = 2\Pr(\boldsymbol{U}_I < \boldsymbol{U}_I^*) = 2\Pr\left\{\operatorname{sign}(\boldsymbol{U}_I^* - \boldsymbol{U}_I) = \mathbf{1}\right\}$$

- ✓ Set $\boldsymbol{Y} = (1/2) \times \{\operatorname{sign}(\boldsymbol{U}_I^* - \boldsymbol{U}_I) + \mathbf{1}\}$ and observe that $\boldsymbol{Y}$ is palindromic.
- ✓ The concordance signature $\boldsymbol{\kappa}_C$ uniquely determines the law of $\boldsymbol{Y}$.
- ✓ The law of $\boldsymbol{Y}$ uniquely determines an extremal mixture C^{E}, viz.

$$w_k = (1/2) \times \Pr(\boldsymbol{Y} = \boldsymbol{s}_k), \quad k \in \{1, \dots, 2^{d-1}\}$$

- ✓ We find that $\boldsymbol{\kappa}_C = \boldsymbol{\kappa}_{C^{\mathrm{E}}}$. □

Attainability of Kendall rank correlation matrices

Let $P^{(k)}$ be the extremal correlation matrix of the kth extremal copula, viz.

$$P^{(k)} = (2\boldsymbol{s}_k - \mathbf{1})(2\boldsymbol{s}_k - \mathbf{1})^\top,$$

where $k \in \{1, \dots, 2^{d-1}\}$.

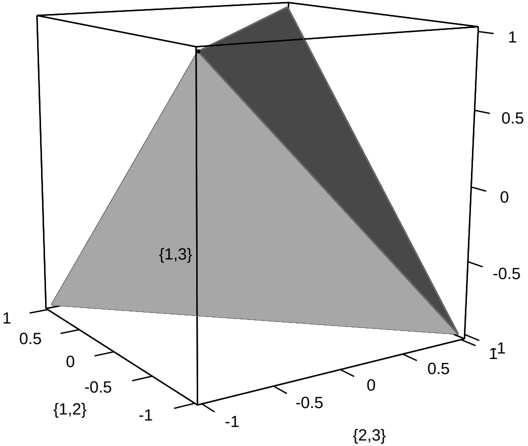
Characterization of Kendall's tau matrices

The $d \times d$ correlation matrix P is a Kendall's tau rank correlation matrix if and only if P can be represented as a convex combination of the extremal correlation matrices in dimension d, i.e.,

$$P = \sum_{k=1}^{2^{d-1}} w_k P^{(k)} \; .$$

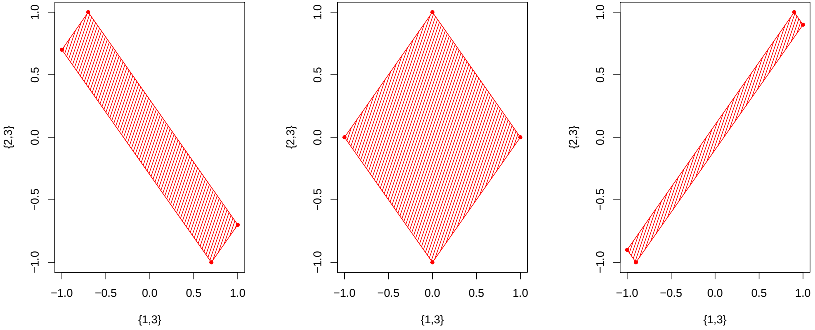
The convex hull of the matrices $P^{(k)}$ is called the cut-polytope (Huber and Marić, 2019).

Illustration in 3D



The cut-polytope of $(\tau_{\{1,2\}}, \tau_{\{1,3\}}, \tau_{\{2,3\}})$.

Illustration in 2D



Set of attainable values of $(\tau_{\{1,3\}}, \tau_{\{2,3\}})$ when $\tau_{\{1,2\}}$ is equal to -0.7 (left), 0 (middle), and 0.9 (right).

Not all correlation matrices are attainable

Consider the matrix

$$\frac{1}{12}\begin{pmatrix} 12 & -5 & -5 \\ -5 & 12 & -5 \\ -5 & -5 & 12 \end{pmatrix}.$$

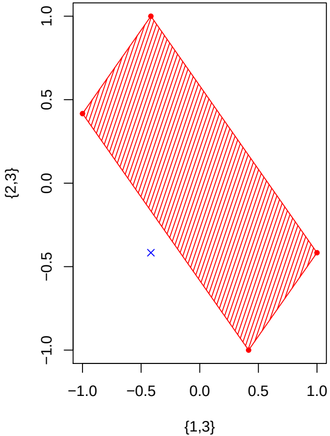
This matrix is symmetric and positive definite, with eigenvalues

$$(17/12, 17/12, 1/6).$$

However, it is not a Kendall's rank correlation matrix. The only possible weights would be

$$\frac{1}{24} \times (-27, 17, 17, 17)$$

A picture to convince you



Attainability of higher-order concordance probabilities

Let $\boldsymbol{\kappa}$ be the even concordance signature of a copula C.

Let A_d be the $2^{d-1} \times 2^{d-1}$ matrix with columns $\boldsymbol{a}^{\mathcal{E}}_{J_k}$, $k \in \{1, \ldots, 2^{d-1}\}$, i.e., the even concordance signatures of the extremal copulas.

The fact that $\boldsymbol{\kappa}$ is also the even concordance signature of a unique extremal copula means that the constrained linear equation system

$$\boldsymbol{\kappa} = A_d \boldsymbol{w}, \quad \boldsymbol{w} \geq \mathbf{0}$$

has a unique solution.

Aside: It can be shown that A_d is of full rank.

Example

Consider C in $d = 4$ and suppose that all bivariate tau's are equal, viz.

$$\boldsymbol{\kappa}_C = c(1, \underbrace{\kappa_2, \ldots, \kappa_2}_{6\times}, \kappa_4).$$

In principle, for any I with $|I| = 2$,

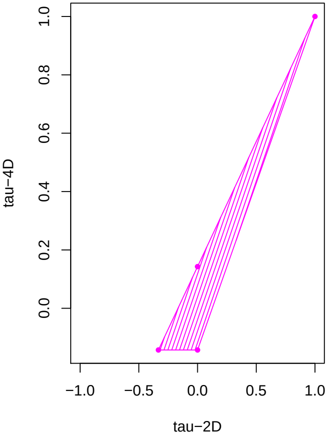
$$\tau_I \equiv \tau_2 \in [-1, 1] \quad \text{and} \quad \tau_{\{1,2,3,4\}} \equiv \tau_4 \in [-1/7, 1].$$

With the results we just derived, we can show that in fact,

$$\kappa_2 \in [1/3, 1] \quad \text{and} \quad \kappa_4 \in [\max(2\kappa_2 - 1, 0), (3\kappa_2 - 1)/2],$$

which restricts the set of attainable pairs (τ_2, τ_4).

Visualization of the set (τ_2, τ_4)



This result holds whatever the dependence structure!

Compatibility problems

Let $S \subset \mathcal{E}(\mathcal{D})$ be a strict subset of the even power set containing $\emptyset$ and

$$\boldsymbol{\lambda} = (\lambda_I : I \in S)$$

be some corresponding candidate partial concordance signature.

Questions to be asked:

- ✓ Is $\boldsymbol{\lambda}$ part of a signature of a d-variate continuous distribution?
- ✓ If it is, can we identify the remaining part of the signature compatible with $\boldsymbol{\lambda}$?

This is related to the harder compatibility problem: Are some fixed lower-dimensional copulas margins of a bona-fide d-variate copula?

Finding compatible signatures

Let $A_d^{(1)}$ be the matrix consisting of the rows of A_d that correspond to S.

A linear programming problem

Find the set of all $\boldsymbol{w} \in \mathbb{R}^{2^{d-1}}$ such that

$$A_d^{(1)} \boldsymbol{w} = \boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{w} \geq \boldsymbol{0}.$$

If the set is non-empty, it is a convex polytope with vertices $\boldsymbol{w}_i$, $i \in \{1, \ldots, m\}$ that can be found using the algorithm of Avis and Fukuda.

The remaining part of the signature is then any element of the convex hull

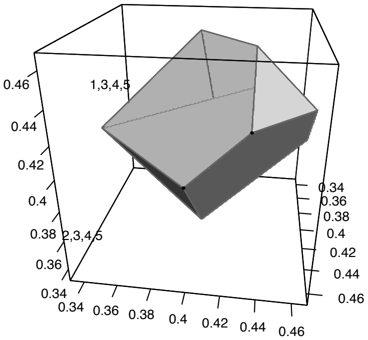
$$\left\{ \sum_{i=1}^{m} \alpha_i A_d^{(2)} \boldsymbol{w}_i, \ \sum_{i=1}^{m} \alpha_i = 1, \ \alpha_1 \geq 0, \ldots, \alpha_m \geq 0 \right\} .$$

Example

Take $d = 5$ and suppose that

$$\kappa_{\{i,j\}} = 2/3,\ i \neq j \quad \text{and} \quad \kappa_{\{1,2,3,4\}} = \kappa_{\{1,2,3,5\}} = 0.4.$$

To complete $\boldsymbol{\kappa}$, we need to specify $\kappa_{\{1,2,4,5\}}$, $\kappa_{\{1,3,4,5\}}$ and $\kappa_{\{2,3,4,5\}}$.



Finding a concrete compatible concordance signature

If we wish to find one compatible concordance signature, we can also solve

An optimization problem

Minimize

$$\|A_d^{(2)}\boldsymbol{w}\| \quad \left(\text{or } \|\mathbf{1} - A_d^{(2)}\boldsymbol{w}\|\right)$$

subject to the constraints

$$A_d^{(1)}\boldsymbol{w} = \boldsymbol{\lambda} \quad \text{and} \quad \boldsymbol{w} \geq \mathbf{0}.$$

In this example, we get

$$\kappa_{\{1,2,4,5\}} = \kappa_{\{1,3,4,5\}} = \kappa_{\{2,3,4,5\}} = \frac{17}{45} \approx 0.378 \quad \left(\text{or } \frac{4}{9} \approx 0.444\right).$$

Elliptical copulas seem versatile

- ✓ The copula of an elliptical random vector $\boldsymbol{X}$ with $\Pr(\boldsymbol{X} = \mathbf{0}) = 0$ is called an elliptical copula.
- ✓ Any correlation matrix is the correlation matrix of an elliptical (in fact Normal) distribution
- ✓ Elliptical copulas are parametrized by a correlation matrix P.
- ✓ From Lindskog et al. (2002),

$$\tau_{\{i,j\}} = \frac{2}{\pi} \arcsin(P_{ij}), \quad i \neq j \in \{1, \ldots, d\}.$$

- ✓ It is tempting to believe that any Kendall rank correlation matrix is attainable within the class of elliptical copulas.

Surprise!

Consider the matrix

$$M = \begin{pmatrix} 1 & -0.19 & -0.29 & 0.49 \\ -0.19 & 1 & -0.34 & 0.30 \\ -0.29 & -0.34 & 1 & -0.79 \\ 0.49 & 0.30 & -0.79 & 1 \end{pmatrix}.$$

Solving the optimization problem discussed before gives the weights

$$\boldsymbol{w} = (0.04, 0.005, 0.36, 0, 0.0625, 0.2475, 0.2825, 0.0025).$$

This means that M is a Kendall's rank correlation matrix.

But $\sin(\pi M/2)$ is not positive semi-definite.

Hence M is not a Kendall's rank correlation matrix of an elliptical copula.

Estimating concordance signatures from data

- ✓ Consider a random sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ from an unknown distribution with copula C and continuous marginals $F_1, \dots, F_d$.

- ✓ For any $I \subseteq \{1, \dots, d\}$, $|I| \geq 2$, a natural estimator of κ_I is

$$\kappa_{I,n} = \frac{2}{n(n-1)} \sum_{i \neq j} \prod_{k \in I} \mathbf{1}\{X_{ik} \leq X_{jk}\}.$$

- ✓ $\kappa_{I,n}$ corresponds to the well-known estimators of Kendall's tau for $d = 2$ (Hoeffding, 1948) and $d > 2$ (Genest et al., 2011).

- ✓ Asymptotic normality is easy to establish using the U-statistics theory.

One last result

Theorem 4: Empirical signatures are intrinsic

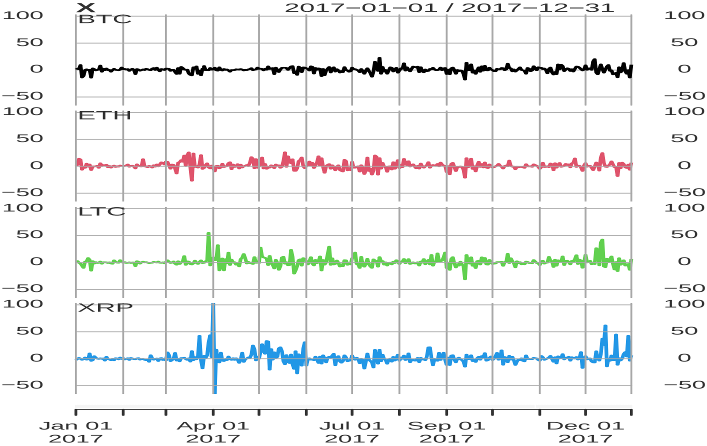
Assuming that $n \geq 2$ and there are no ties in the sample, there exists a d-dimensional copula C_n such that $(\kappa_{I,n}, I \subseteq \mathcal{P}(\mathcal{D}))$ is the concordance signature of C_n.

Idea behind: Let $\boldsymbol{Y}_{ij} = \left\{ \text{sign}(\boldsymbol{X}_i - \boldsymbol{X}_j) + 1 \right\}/2$ for $i \neq j$ and set

$$\widehat{w}_k = \frac{2}{n(n-1)} \sum_{i<j} \left(I_{\{\boldsymbol{Y}_{ij}=\boldsymbol{s}_k\}} + I_{\{\boldsymbol{Y}_{ij}=\boldsymbol{1}-\boldsymbol{s}_k\}} \right).$$

Then $(\widehat{w}_k, k = 1, \ldots, 2^{d-1})$ are the weights of the extremal mixture with concordance signature $(\kappa_{I,n}, I \subseteq \mathcal{P}(\mathcal{D}))$.

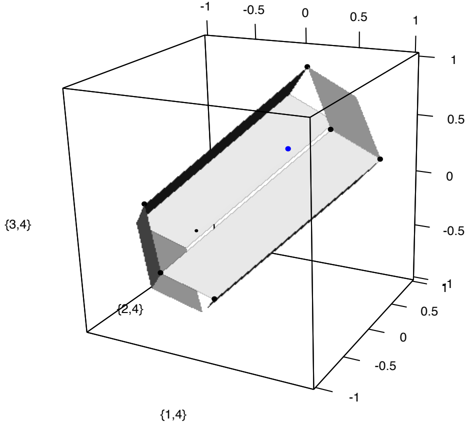
Illustration with crypto currencies



Log-returns on Bitcoin, Ethereum, Litecoin and Ripple prices in USD.

Range of unobserved correlations

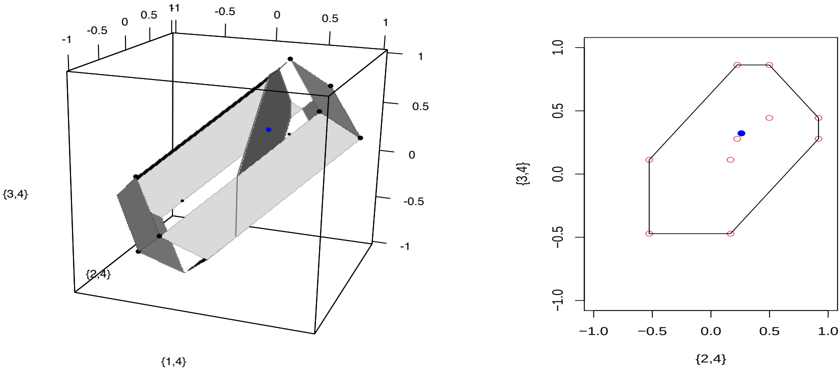
Imagine we have not observed Ripple prices and we only know Kendall's tau between Bitcoin, Ethereum and Litecoin: $(0.278, 0.333, 0.361)$.



Range of attainable correlations with Ripple.

Adding more information

Now suppose we are being told that the correlation between Ripple and Bitcoin is 0.196.



Range of attainable correlations with Ripple.

Thank you for your attention!

On attainability of Kendall's tau matrices and concordance signatures

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https://arxiv.org/abs/2009.08130

The package KendallSignature on GitHub

https://github.com/ajmcneil/KendallSignature

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Elliptical copulas are actually quite restrictive...

It's worse when we look at the entire concordance signature.

Take any $I \subset \{1, \ldots, d\}$, $|I| \geq 2$. From the stochastic representation of $\boldsymbol{X}$,

$$\begin{aligned}
\kappa_I(\boldsymbol{X}) &= 2\Pr\left(\boldsymbol{X}_I - \boldsymbol{X}_I^* < \mathbf{0}\right) \\
&= 2\Pr\left\{(R_1 A\boldsymbol{S})_I < \mathbf{0}\right\} = 2\Pr\left\{(R_2 A\boldsymbol{S})_I < \mathbf{0}\right\} \\
&= 2\Pr(\boldsymbol{X}_I < \mathbf{0}) = 2\Pr(\boldsymbol{Z}_I < \mathbf{0}) = \kappa_I(\boldsymbol{Z}),
\end{aligned}$$

where $\boldsymbol{Z}$ is multivariate Normal with the same correlation matrix P. Hence

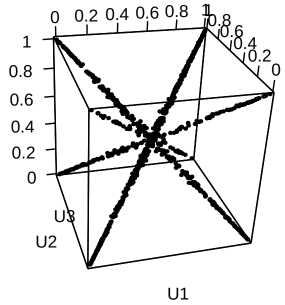
$$\boldsymbol{\kappa}_{\boldsymbol{X}} = \boldsymbol{\kappa}_{\boldsymbol{Z}}.$$

Once the bivariate concordance probabilities of an elliptical copula have been determined, all higher order κ_I's have been fixed also.

The spider copula theorem

Theorem 3: Spider copula theorem

As $\nu \to 0$ the d-dimensional Student t copula $C^t_{\nu,P}$ converges pointwise to the unique extremal mixture copula that shares its concordance signature.



Scatterplot of data with distribution $C^t_{\nu,P}$ when $\nu = 0.03$ and P is the 3×3 matrix with elements $\rho_{12} = 0.2$, $\rho_{13} = 0.5$ and $\rho_{23} = 0.8$.