Optimal annuitisation, housing and reverse mortgage in retirement in the presence of a means-tested public pension

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- J.G. Andréasson and P.V. Shevchenko (2022). **A bias-corrected Least-Squares Monte Carlo for solving multi-period utility models.** *European Actuarial Journal* 12, pp. 349–379.
- J.G. Andréasson and P.V. Shevchenko (2024). **Optimal annuitisation, housing and reverse mortgage in retirement in the presence of a means-tested public pension.** *European Actuarial Journal*. DOI: 10.1007/s13385-024-00379-3.
- J.G. Andreasson, P.V. Shevchenko, and A. Novikov (2017), **Optimal Consumption, Investment and Housing with Means-tested Public Pension in Retirement.** *Insurance: Mathematics and Economics* 75, 32-47.
- J.G. Andréasson and P.V. Shevchenko (2017). **Assessment of Policy Changes to Means-Tested Age Pension Using the Expected Utility Model: Implication for Decisions in Retirement.** *Risks 5*, 47:1-47:21.

Mathematical Problem Definition

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a filtered complete probability space and $\mathcal{F}_t$ represents the information available up to time t. All the processes introduced below are adapted to $\{\mathcal{F}_t\}_{t\geq 0}$.

Notation:

- Controlled state variable $X = (X_t)_{t=t_0,\dots,T}$
- Control $\pi = (\pi_t)_{t=t_0,\dots,T}$
- Random disturbance $Z = (Z_t)_{t=t_0,\dots,T}$
- State variable evolution $X_{t+1} = T(X_t, \pi_t, Z_{t+1})$

Objective: maximise the expected value of the total reward

$$V_{t_0}(x) = \sup_{\pi} \mathbb{E}\left[\beta^{T-t_0} G_T(X_T) + \sum_{t=t_0}^{T-1} \beta^{t-t_0} R_t(X_t,\pi_t) | X_{t_0} = x\right]$$

where G_T and R_t are functions satisfying integrability conditions and β is discounting factor.

Mathematical Problem Definition

This type of problem can be solved with backward recursion of the Bellman equation, where

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\begin{aligned}
V_T(x) &= G_T(x), \\
V_t(x) &= \sup_{\pi_t} \left\{ R_t(x, \pi_t) + \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) | X_t = x; \pi_t\right] \right\}.
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Optimal value of control is found as

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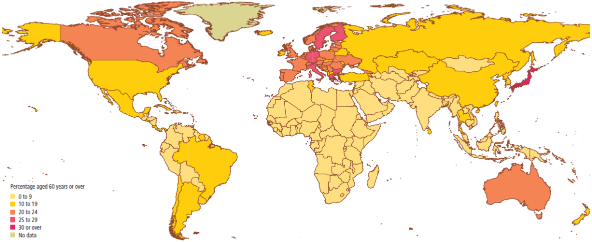
\pi_t^*(x) = \arg \sup_{\pi_t \in \mathcal{A}_t} \left\{ R_t(x, \pi_t) + \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) \,\middle|\, X_t = x; \pi_t\right] \right\}.
$$

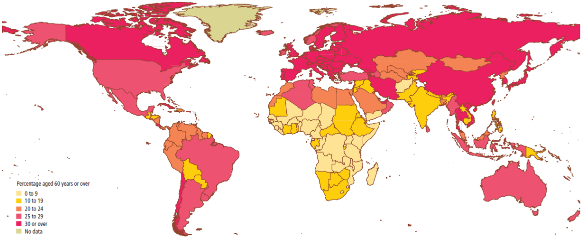
The solution of such problem is often not possible to find analytically and numerical methods are required. As the number of state variables, stochastic processes, or control variables increases, the numerical solution becomes very expensive computationally and simulation methods such as LSMC (Longstaff and Schwartz, 2001; Tsitsiklis and Van Roy, 2001) are favoured.

Population Aging - United Nations (2013): percentage aged 60 years or over, 2012 vs 2050 forecast

2012



2050



Motivation

- Australia's accumulation benefit pension system is still young, but superannuation assets already accumulated \$2.7tn in June 2018 (\$3.5tn in March 2024, 4th largest in the world).
- More retirees - due to both increased life expectancy and an ageing population. Currently 15% of population is 65+ people.
- Age and Service Pension payments will change from 2.9% of GDP in 2015 to 3.6% in 2055 (the number of 65+ will more than double).
- Social security and welfare is 38% of taxpayers money in 2018-19 Australian government budget (and approximately the same in 2022-23), where assistance for aged Australians is the largest part.
- Limited knowledge amongst retirees (and advisors) to manage funds and Age Pension (Spicer et al., 2013). Retirement funds have changed from monthly benefit to lump sum at retirement.
- **Modelling consumption, bequest, home ownership, and investment is important for retirees and the Australian Pension system. Current modelling typically is scenario based and ignores the dynamic response or too simplistic.**

Australian superannuation

- Three pillars - **superannuation guarantee, private savings and government provided Age Pension**.
- Superannuation guarantee contribution rate: is 9% in 2002-03 increasing to 11% in 2023-2024 and set to reach 12% in 2025.
- Means-tested Age Pension: subject to income-test and asset-test, and entitlement age of 65.5 in 2017 (increased to 67 from 2023).
- **Family home is excepted from Age pension asset-test.**
- Income-test based on actual income, deemed income and drawdown of allocated pension accounts.
- Policies and regulations are constantly changing.
- **Allocated pension accounts** are purchased with superannuation, and subject to **minimum withdrawal rates**.

Age	$\leq$64	65–74	75–79	80–84	85–89	90–94	$\geq$95
Min. drawdown	4%	5%	6%	7%	9%	11%	14%

	PRE2015	2015	2017
Full Age Pension singles ($P^{\mathrm{S}}_{\mathrm{max}}$)	\$22,721	\$22,721	\$22,721
Full Age Pension couples ($P^{\mathrm{C}}_{\mathrm{max}}$)	\$34,252	\$34,252	\$34,252
Income-Test	Drawdown	Deemed	Deemed
Threshold singles (L^{S}_{I})	\$4264	\$4264	\$4264
Threshold couples (L^{C}_{I})	\$7592	\$7592	\$7592
Rate of reduction (ϖ^{d}_{I})	\$0.5	\$0.5	\$0.5
Deeming threshold singles (κ^{S})	-	\$49,200	\$49,200
Deeming threshold couples (κ^{C})	-	\$81,600	\$81,600
Deeming rate below κ^{d} (ς_{-})	-	1.75%	1.75%
Deeming rate above κ^{d} (ς_{+})	-	3.25%	3.25%
Asset-Test			
Threshold homeowners singles ($L^{S,h=1}_{\mathrm{A}}$)	\$209,000	\$209,000	\$250,000
Threshold homeowners couples ($L^{C,h=1}_{\mathrm{A}}$)	\$296,500	\$296,500	\$375,000
Threshold non-homeowners singles ($L^{S,h=0}_{\mathrm{A}}$)	\$360,500	\$360,500	\$450,000
Threshold non-homeowners couples ($L^{C,h=0}_{\mathrm{A}}$)	\$448,000	\$448,000	\$575,000
Rate of reduction (ϖ^{d}_{A})	\$0.039	\$0.039	\$0.078

In our study we use the Australian pension system rules from 2017 as in Andréasson et al. (2017). However, note that the pension system rules are revised regularly.

Expected Utility Model: Base Model Assumptions

- Agent (household) is an expected utility maximiser based on **hyperbolic absolute risk aversion (HARA)**.
- Assuming time-separable additive utility functions for consumption, housing and bequest. Each year alive the retiree receives utility from consumption and housing, and bequest in case of death.
- Start at retirement $t = t_0$ where the retiree can allocate wealth into **housing and an allocated pension account**. Lives no longer than terminal time T.
- Starts as either a couple or single. Couples have mortality risk where if one spouse dies, it becomes a single household.
- **All wealth is held in an allocated pension account, which does not attract taxes on capital gains.**
- Each period the retiree receives Age Pension, consumes part of his wealth and allocates the remaining into a risky asset and risk-free asset.

Base Model Definition

Denote a **state vector** as $X_t = (W_t, G_t, H)$, where

- W_t denotes the current level of wealth,
- G_t is life status with realisations in $\mathcal{G} = \{\Delta, 0, 1, 2\}$ indicating whether the agent is dead, died this period, alive in a single household or alive in a couple household,
- H denotes wealth invested in housing at t_0.

The utility received at time t is subject to the **control variables**

- α_t - proportion drawdown of liquid wealth,
- δ_t - proportion liquid wealth allocated to risky assets,
- ϱ - wealth allocated to housing only at time $t = t_0$.

We aim to find optimal **decision rule** $\pi_t(x_t) = (\alpha_t, \delta_t)$ which is the action at time t and depends on the current state x_t. Then a sequence (policy) of decision rules is given by $\pi = (\pi_{t_0}, ..., \pi_{T-1})$ for $t = t_0, ..., T-1$.

Model Definition

Wealth process is driven by stochastic return $Z_{t+1} \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma^2)$ and deterministic risk-free rate r, and controlled by drawdown α_t and risky asset allocation δ_t, given by

$$W_{t+1} = (W_t - \alpha_t W_t)\left(\delta_t e^{Z_{t+1}} + (1-\delta_t)e^{r_t}\right),$$

$$\begin{aligned} \text{s.t} \quad & C_t = \alpha_t W_t + P_t, \\ & W_t + P_t - C_t \geq 0, \\ & W_{t_0} = \mathrm{W} - H, \\ & H \in \{0, [H_L, \mathrm{W}]\}, \end{aligned}$$

where W_t is the liquid wealth before withdrawal, r_t is the time dependent but deterministic **real** risk-free rate and W is the initial total wealth.

Model Definition

Each period the agent receives utility, given by

$$R_t(W_t, G_t, \alpha_t, H) = \begin{cases} U_C(C_t, G_t, t) + U_H(H, G_t), & \text{if } G_t = 1, 2, \\ U_B(W_t, H), & \text{if } G_t = 0, \\ 0 & \text{if } G_t = \Delta, \end{cases}$$

with terminal condition ($t = T$) given by

$$\widetilde{R}(W_T, G_T, H) = \begin{cases} U_B(W_T, H), & \text{if } G_T \geq 0 \\ 0, & \text{if } G_T = \Delta. \end{cases}$$

We need to find a solution of the following problem

$$\widetilde{V} := \max_{\varrho} \left[\sup_{\pi} \mathbb{E}^{\pi}_{t_0} \left[\beta_{t_0,T} \widetilde{R}(W_T, G_T, H) + \sum_{t=t_0}^{T-1} \beta_{t_0,t} R_t(W_t, G_t, \alpha_t, H) \right] \right]$$

where $\mathbb{E}^{\pi}_{t_0}[\cdot]$ is the expectation conditional on information and decision at time $t = t_0$ and $\beta_{t,t'}$ is the discounting from t to t'.

Model Definition

Consumption is based on drawdown of wealth and Age Pension received. Utility is received from consumption exceeding the consumption floor.

$$U_C(C_t, G_t, t) = \frac{1}{\psi^{t-t_0}\gamma_d}\left(\frac{C_t - \bar{c}_d}{\zeta_d}\right)^{\gamma_d}, d = \begin{cases} \mathrm{C}, & \text{if } G_t = 2 \quad \text{(couple)}, \\ \mathrm{S}, & \text{if } G_t = 1 \quad \text{(single)}, \end{cases}$$

where $\gamma_d \in (-\infty, 0)$ is the risk aversion, $\bar{c}_d$ the consumption floor, C_t the consumption for year t and ζ_d the scaling factor to normalise between singles and couples. Let $\psi \in [1, \infty)$ be the utility parameter for the health proxy.

Model Definition

Bequest utility function is defined as

$$U_B(W_t, H) = \left(\frac{\theta}{1-\theta}\right)^{1-\gamma_{\mathrm{S}}} \frac{\left(\frac{\theta}{1-\theta}a + W_t + H\right)^{\gamma_{\mathrm{S}}}}{\gamma_{\mathrm{S}}},$$

where W_t is the liquid wealth available for bequest, γ_{S} the risk aversion of bequest utility (same as consumption risk aversion for singles), a is the threshold for luxury bequest and $\theta \in [0, 1)$ is the degree of altruism.

Model Definition

Housing generates utility through a flow of services, approximated with the house value,

$$U_H(H) = \frac{1}{\gamma_{\mathrm{H}}} \left(\frac{\lambda_d H}{\zeta_d} \right)^{\gamma_{\mathrm{H}}},$$

where γ_H is the risk aversion parameter for housing (allowed to be different from risk aversion for consumption and bequest), ζ_d is the same scaling factor as for consumption, H is the market value of the family home at time of purchase and $\lambda_d \in [0, 1]$ is the preference of housing defined as a proportion of the market value.

Age Pension Formula

Over 90% of income comes from allocated pensions, hence we assume that wealth in the asset test equals allocated pension and the drawdown is considered income. The means test is subject to different thresholds for single, couples and whether they are homeowners or not, where

- P^d_{max} is the full Age Pension.
- L^d is the threshold for the asset/income test.
- ϖ^d is the taper rate for assets/income test.
- $h = \{0, 1\}$ whether the retiree is a homeowner or not.

Combined Age Pension formula

$$P_t := f(\alpha_t, W_t, t) = \max\left[0, \min\left[P^d_{\max}, \min\left[P_{\mathrm{A}}(W_t), P_{\mathrm{I}}(\alpha_t W_t, t)\right]\right]\right].$$

- Asset test: $P_{\mathrm{A}}(W_t) = P^d_{\max} - (W_t - L^d_{\mathrm{A}})\varpi^d_{\mathrm{A}}$.
- Income test: $P_{\mathrm{I}}(\alpha_t W_t, t) = P^d_{\max} - (\alpha_t W_t - M(t) - L^{d,h}_{\mathrm{I}})\varpi^d_{\mathrm{I}}$
- Income test deduction: $M(t) = \frac{W_{t_0}}{e_{t_0}}(1 + \widetilde{r})^{t_0 - t}$, where e_{t_0} is the life expected at age t_0 and $\widetilde{r}$ the inflation.

New Age Pension policy

The calibrated model is already outdated:

- From 2015, deemed income is used in allocated pension (previously drawdown).
- In 2017, the asset-test thresholds were ‘rebalanced’ and taper rate doubled.

Deemed income in the pension function:

$$P_{\mathrm{I}} := P^{d}_{\max} - \left(P_{\mathrm{D}}(W_t) - L^{d}_{\mathrm{I}}\right) \varpi^{d}_{\mathrm{I}},$$

$$P_{\mathrm{D}}(W_t) = \varsigma_{-} \min\left[W_t, \kappa^{d}\right] + \varsigma_{+} \max\left[0, W_t - \kappa^{d}\right].$$

Stochastic control problem formulation

Optimal stochastic control problem is solved numerically with backwards recursion of the Bellman equation, starting from terminal condition

$$V_T(X_T) = \widetilde{R}_T(W_T, G_T, H),$$

and for each $t < T$

$$V_t(X_t) = \sup_{\pi_t} \left\{ R_t(W_t, G_t, \alpha_t, H) + \beta_{t,t+1} \, \mathbb{E}_t \left[V_{t+1}(X_{t+1}) \mid X_t; \pi_t \right] \right\}.$$

Then, optimal housing decision control ϱ maximising $V_0(X_0)$ is calculated.

Note that the death probabilities are not explicit in the objective function, but affect the evolution of the family status and are involved in the calculation of the conditional expectation.

Numerical solution

- Discretise wealth state W and house state H on log-equidistant grid and solve recursively with backwards induction.
- Family status state G can be avoided by weighting the reward function with survival probabilities in the value function.
- Numerical integration by Gauss-Hermite Quadrature, with 5 nodes.
- Interpolation via shape preserving Piecewise Cubic Hermite Interpolation Polynomial (PCHIP), which preserves the monotonicity and concavity.
- Decision variable for housing enough to solve at $t = t_0$.

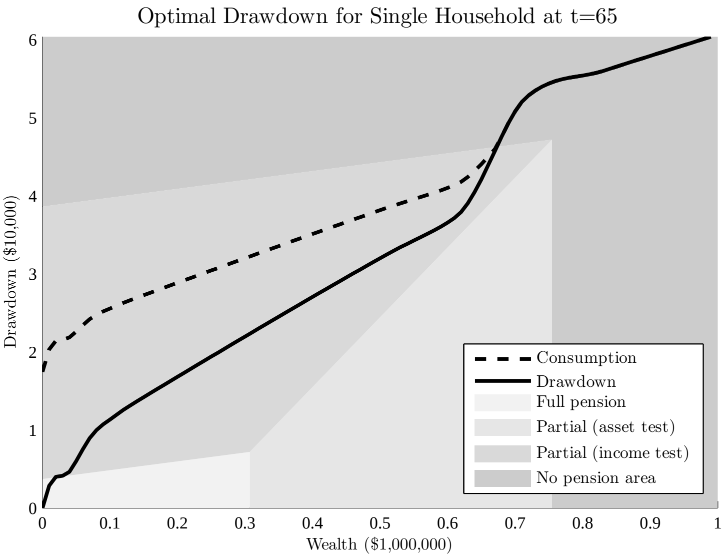
Calibration

- Data were taken from Australian Bureau of Statistics Household Expenditure Survey (HES) 2009-2010, and Survey of Income and Household (SIH) 2009-2010.
- Only a snapshot, does not offer data of cohorts over time.
- Data aggregated based on households in retirement and not part of the work force, split over both single (2,038 data points) and couple households (2,017 data points).
- Calibrate parameters via maximum likelihood estimation on consumption and housing samples.

Calibrated utility parameters with standard error:

	γ_S	γ_C	γ_H	θ	a	$\bar{c}_S$	$\bar{c}_C$	ψ	λ
Value	- 2.77	- 2.29	-2.58	0.54	26 741	11 125	18 970	1.47	0.037
Std. Error	0.12	0.14	0.19	0.03	1 377	1 011	1 682	0.04	0.006

Calibration Output - Optimal Consumption



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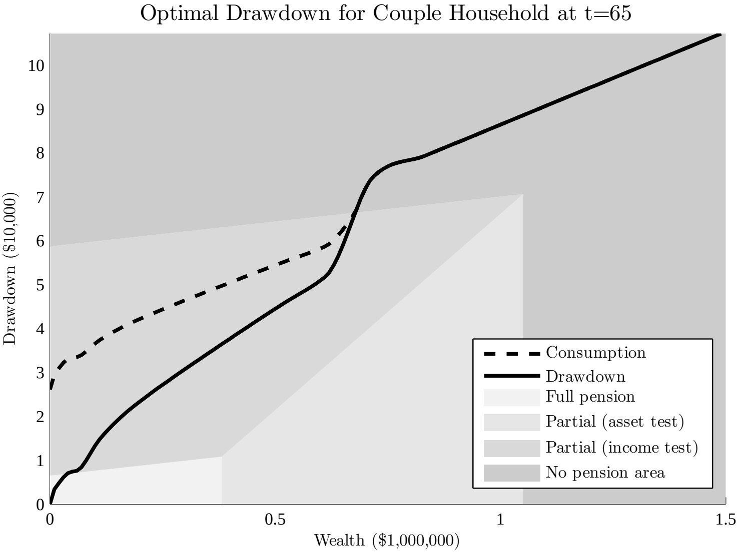

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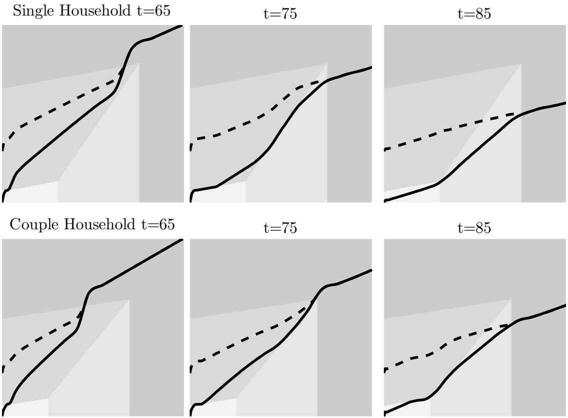


Figure: Comparison of optimal consumption over time.

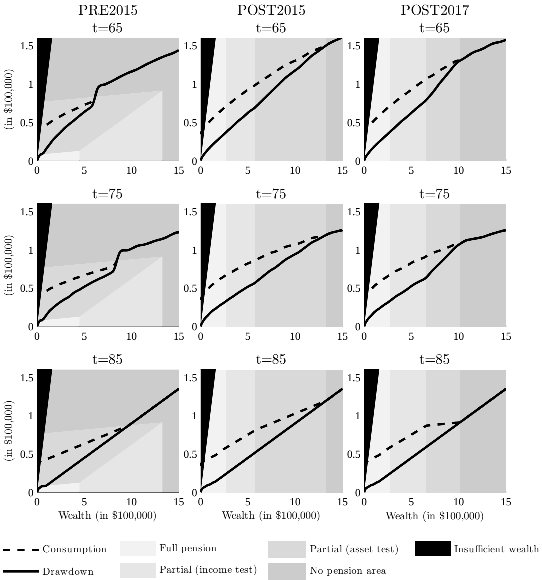


Figure: Optimal drawdown and consumption for non-homeowner couple households for a given liquid wealth at the age t, under the three different policy scenarios in the case of low returns ($\mu = 0.0325$).

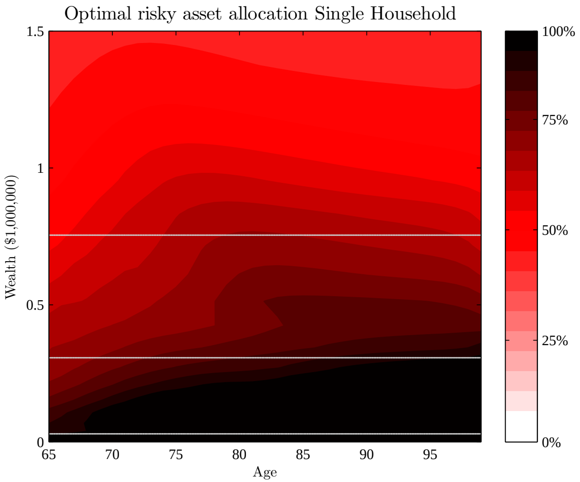


Figure: Optimal allocation of risky assets for single non-homeowners. The horizontal lines (from bottom up) show the threshold a, the threshold for partial Age Pension due to asset test, and the threshold for no Age Pension due to asset test.

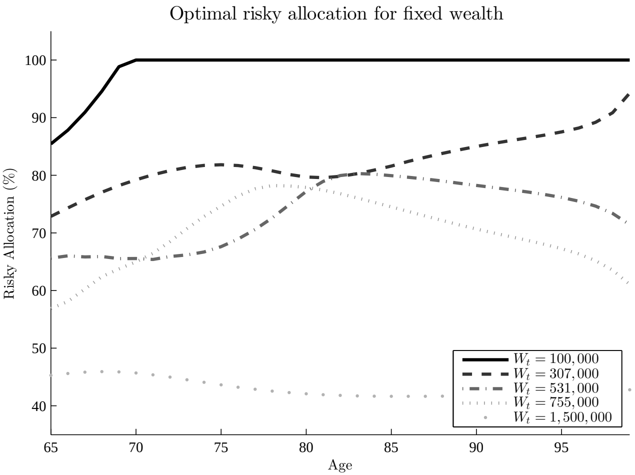


Figure: Optimal allocation of risky assets for single non-homeowners, given fixed wealth. The wealth levels correspond to full pension, the lower threshold of the asset test, partial pension due to asset test, the upper threshold of the asset test, and no pension.

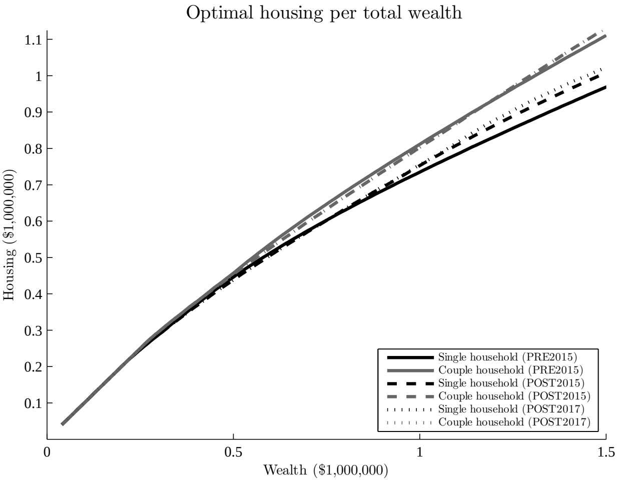


Figure: Optimal housing allocation given by total initial wealth W for single and couple households, under the three policy scenarios with the low return ($\mu = 0.0325$).

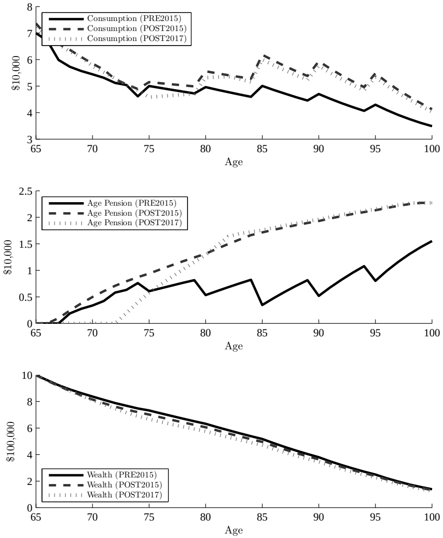
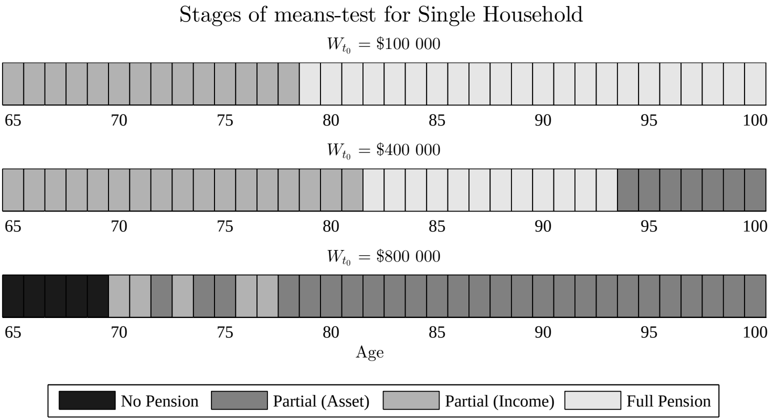


Figure: Comparison of consumption, Age Pension and wealth over a retiree's lifetime with the three different policy scenarios. The retiree starts with \$1m liquid wealth, which grows with the low expected return each year ($\mu = 0.0325$), and drawdown follows the optimal drawdown paths under each policy.

Calibration Output - Phases of Means-test



Conclusions - Calibrated model

- Optimal drawdown is highly sensitive to the means test early in retirement due to the number of expected years remaining to receive Age Pension, but decreases with time so optimal consumption becomes approximately linear.
- **The Age Pension works as a buffer against investment losses, hence optimal allocation to risky asset increases rapidly when the asset test binds and suggest 100% risky allocation when full Age Pension is received.**
- Optimal housing is similar between single and couple households in terms of proportion of wealth, but house value will differ due to different wealth levels. The high allocation for lower wealth matches the characteristics where households with lower wealth levels tend to have the family home as their only asset.

Least-Squares Monte Carlo method for model extensions

The model should be extended with additional deposit account, stochastic interest rate, housing decisions, reverse mortgage, annuitization, ...

- Additional states and stochastic variables make a quadrature based numerical solution computationally infeasible.
- Least-Squares Monte Carlo (LSMC) is an approximate method for solving stochastic control problems, e.g. Longstaff and Schwartz (2001) for valuation of American options.
- Essentially a simulation and regression algorithm, where random paths are simulated and the conditional expectation in Bellman equation is approximated with a regression function, then solved via backwards recursion for stochastic control problems.
- Original exogenous LSMC extended in Kharroubi et al. (2014) with endogenous state variables and control randomisation.

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$$

The solution of such problem is often not possible to find analytically and numerical methods are required.
As the number of state variables, stochastic processes, or control variables increases, the numerical solution becomes very expensive computationally and simulation methods such as LSMC are favoured.

- If the state variable is not affected by the control, the idea behind utilitising the LSMC method is to approximate the conditional expectation

$$\Phi_t(X_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1})|X_t\right],$$

by a regression scheme with independent variables X_t, and response variable $\beta V_{t+1}(X_{t+1})$. The approximation of the function is then denoted as $\widehat{\Phi}_t$.

- If the state variable is affected by control, then techniques such as control randomization are required where the conditional expectation

$$\Phi_t(X_t, \pi_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1})|X_t; \pi_t\right]$$

is estimated by regression of $\beta V_{t+1}(X_{t+1})$ on X_t and randomised π_t Kharroubi et al. (2014).

Arguments for LSMC

Arguments *for* LSMC:

- Does not suffer from "curse of dimensionality", hence faster than other numerical methods as the number of state variables increase.
- No restrictions on dynamics of stochastic processes (contrary to PDE's). Enough to be able to simulate a path.
- Parametric estimate in feedback form of control (no grid required).

Arguments *against* LSMC:

- Approximate method only, and can have substantial errors piling up over multiple periods.
- Can be computationally intensive, especially for the optimisation of control variables.
- Basis function can be difficult to find and is highly problem specific.

LSMC for models with Utility Functions

There are difficulties with LSMC in the case of utility type models; difficult to fit due to extreme curvature over the full sample (extreme heteroskedasticity). Proposed method: regressing on the *transformed* value function and adjusting for the retransformation bias.

Define a transformation H^{-1} such that $H^{-1}(H(x)) = x$. Let $\mathbf{L}(X_t, \pi_t)$ be a vector of basis functions and $\mathbf{\Lambda}_t$ the corresponding regression coefficients vector, such that

$$\mathbb{E}\left[H^{-1}(\beta V_{t+1}(X_{t+1}))|X_t; \pi_t\right] = \mathbf{\Lambda}_t'\mathbf{L}(X_t, \pi_t).$$

If M independent Markovian paths of state and control variables are simulated, one can consider the ordinary linear regression

$$H^{-1}(\beta V_{t+1}(X_{t+1}^m)) = \mathbf{\Lambda}_t'\mathbf{L}(X_t^m, \pi_t^m) + \epsilon_t^m,$$

$$\epsilon_t^m \overset{iid}{\sim} F_t(\cdot), \;\; \mathbb{E}[\epsilon_t^m] = 0, \;\; \mathrm{var}[\epsilon_t^m] = \sigma_t^2, \;\; m = 1, ..., M$$

$$\hat{\mathbf{\Lambda}}_t = \arg\min_{\mathbf{\Lambda}} \sum_m \left[H^{-1}(V(t, X_t^m)) - \mathbf{\Lambda}'\mathbf{L}(X_t^m, \pi_t^m)\right]^2.$$

Duan's Smearing Estimate (Duan, 1983)

Our objective is to estimate $\Phi_t(X_t, \pi_t) = \mathbb{E}[\beta V_{t+1}(X_{t+1})|X_t; \pi_t]$:

$$H^B(\mathbf{\Lambda}_t'\mathbf{L}(X_t, \pi_t)) := \Phi_t(X_t, \pi_t) = \int H(\mathbf{\Lambda}_t'\mathbf{L}(X_t, \pi_t) + \epsilon_t) dF_t(\epsilon_t),$$

where $F_t(\epsilon_t)$ is the distribution of disturbance term ϵ_t. Obviously,

$$\widehat{H}^B(\widehat{\mathbf{\Lambda}}_t'\mathbf{L}(X_t, \pi_t)) = H(\widehat{\mathbf{\Lambda}}_t'\mathbf{L}(X_t, \pi_t))$$

will be neither unbiased nor consistent unless the transformation is linear. If a specific distribution is assumed for ϵ_t, then the integration in can be performed. Otherwise, the empirical distribution of residuals

$$\widehat{\epsilon}_t^m = H^{-1}(\beta V_{t+1}(X_{t+1}^m)) - \widehat{\mathbf{\Lambda}}_t'\mathbf{L}(X_t^m, \pi_t^m),$$

can be used to perform the required integration leading to the following **Smearing Estimate**:

$$\widehat{H}^B(\widehat{\mathbf{\Lambda}}_t'\mathbf{L}(X_t, \pi_t)) = \frac{1}{M}\sum_{m=1}^{M} H(\widehat{\mathbf{\Lambda}}_t'\mathbf{L}(X_t, \pi_t) + \widehat{\epsilon}_t^m),$$

Smearing estimate – example

Suppose we consider regression $\ln Y_i = \boldsymbol{\beta}'\boldsymbol{X}_i + \epsilon_i$ and we want to estimate $\mathbb{E}[Y^\gamma/\gamma]$, then the smearing estimate is

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\left(e^{\widehat{\boldsymbol{\beta}}'\boldsymbol{X}+\widehat{\epsilon}_i}\right)^\gamma}{\gamma} = \frac{\left(e^{\widehat{\boldsymbol{\beta}}'\boldsymbol{X}}\right)^\gamma}{n\gamma}\sum_{i=1}^{n} e^{\widehat{\epsilon}_i\gamma}.$$

The smearing estimate works well for non-normal errors and can accommodate for heteroskedasticity, provided it is not related to a covariate.

Controlled Heteroskedasticity

If heteroskedasticity is present in the regression with respect to state and control variables, a method that accounts for the heteroskedasticity is required. In this case the conditional variance can be modelled as

$$\mathrm{var}[\epsilon_t | X_t, \pi_t] = [\Omega(\boldsymbol{\mathcal{L}}_t' \mathbf{C}(X_t, \pi_t))]^2$$

where $\Omega(\cdot)$ is some positive function, $\boldsymbol{\mathcal{L}}_t$ is the vector of coefficients and $\mathbf{C}(X_t, \pi_t)$ is a vector of basis functions. There are various standard ways to find estimates $\widehat{\boldsymbol{\mathcal{L}}}_t$, the one we use is based on the linear regression of the log of squared residuals $\widehat{\epsilon}_t^m$. Then, one can use the **Smearing Estimate with Controlled Heteroskedasticity**:

$$\widehat{H}^B(\widehat{\boldsymbol{\Lambda}}_t' \mathbf{L}(X_t, \pi_t)) = \frac{1}{M} \sum_{m=1}^{M} H\left(\widehat{\boldsymbol{\Lambda}}_t' \mathbf{L}(X_t, \pi_t) + \Omega(\widehat{\boldsymbol{\mathcal{L}}}_t' \mathbf{C}(X_t, \pi_t)) \frac{\widehat{\epsilon}_t^m}{\Omega(\widehat{\boldsymbol{\mathcal{L}}}_t' \mathbf{C}(X_t^m, \pi_t^m))}\right)$$

Here, it is also common to replace $\widehat{\boldsymbol{\Lambda}}_t$ with the weighted least squares estimator that can be found after estimation of $\Omega(\cdot)$.

Algorithm LSMC for exogenous state

for $t = 1$ **to** N **do** [Forward simulation]
  for $m = 1$ **to** M **do**
    $X_t^m := \mathcal{T}_t(X_{t-1}^m, z_t)$ [simulate path]
  end for
end for
for $t = N$ **to** 0 **do** [Backward solution]
  if $t = N$ **then**
    $\widehat{V}_t(\boldsymbol{X}_t) := R_N(\boldsymbol{X}_t)$
  else if $t < N$ **then**
    [Regress transformed value function on state variables]
    $\widehat{\boldsymbol{\Lambda}}_t := \arg\min_{\boldsymbol{\Lambda}_t} \sum_{m=1}^{M} \left[\boldsymbol{\Lambda}_t' \mathbf{L}(X_t^m) - H^{-1}(\beta \widehat{V}_{t+1}(X_{t+1}^m))\right]^2$
    Find bias corrected transformation $H^B(\widehat{\boldsymbol{\Lambda}}_t' \mathbf{L}(X_t))$
    $\widehat{\Phi}_t(X_t) = H^B(\widehat{\boldsymbol{\Lambda}}_t' \mathbf{L}(X_t))$ [Approximate conditional expectation]
    for $m = 1$ **to** M **do**
      [Optimal control]
      $\pi_t^*(X_t^m) := \arg\sup_{\pi_t \in A} \left\{ R_t(X_t^m, \pi_t) + \widehat{\Phi}_t(X_t^m) \right\}$
      $\widehat{V}_t(X_t^m) := R_t(X_t^m, \pi_t^*(X_t^m)) + \beta \widehat{V}_{t+1}(X_{t+1}^m)$
    end for
  end if
end for

Pricing Bermudan option using LSMC – numerical example

Table: Price and standard error of the Bermudan option estimated using standard LSMC ($\widehat{V}^{(0)}$), LSMC with log transformation of the value function without bias correction ($\widehat{V}^{(1)}$) and LSMC with log transformation of value function and bias correction ($\widehat{V}^{(2)}$) using *smearing estimate*. The results are based on M sample paths and 20 independent repetitions.
The 'exact' price obtained by the Binomial Tree method is $4.3862.

M	$\widehat{V}^{(0)}$	$\widehat{V}^{(1)}$	$\widehat{V}^{(2)}$
1,000	4.4984 (0.032)	4.4336 (0.038)	4.4054 (0.039)
10,000	4.4616 (0.007)	4.4161 (0.007)	4.3962 (0.008)
100,000	4.4457 (0.003)	4.4048 (0.004)	4.3857 (0.004)

LSMC Algorithm: Endogenous state and random control

We utilise a discretised version of Kharroubi et al. (2015) with some modifications in forward simulation.

Algorithm Forward simulation

```
for t = 0 to N − 1 do
    for m = 1 to M do
        [Simulate random samples ]
        X_t^m := Rand ∈ 𝒳                              ▷ State
        π̃_t^m := Rand ∈ 𝒜                              ▷ Control
        z_{t+1}^m := Rand ∈ 𝒵                          ▷ Disturbance
        [Compute the state variable after control]
        X̃_{t+1}^m := 𝒯_t(X_t^m, π̃_t^m, z_{t+1}^m)       ▷ Evolution of state
    end for
end for
```

In LaTeX notation:

for $t = 0$ **to** $N-1$ **do**
- **for** $m = 1$ **to** M **do**
  - [Simulate random samples]
  - $X_t^m := Rand \in \mathcal{X}$ ▷ State
  - $\widetilde{\pi}_t^m := Rand \in \mathcal{A}$ ▷ Control
  - $z_{t+1}^m := Rand \in \mathcal{Z}$ ▷ Disturbance
  - [Compute the state variable after control]
  - $\widetilde{X}_{t+1}^m := \mathcal{T}_t(X_t^m, \widetilde{\pi}_t^m, z_{t+1}^m)$ ▷ Evolution of state
- **end for**

end for

Algorithm Backward solution (Realised value)

for $t = N$ **to** 0 **do**
  if $t = N$ **then** $\widehat{V}_t(\widetilde{\boldsymbol{X}}_t) := R_N(\widetilde{\boldsymbol{X}}_t)$
  else if $t < N$ **then**
    [Regression of transformed value function]
    $\widehat{\boldsymbol{\Lambda}}_t := \arg\min_{\boldsymbol{\Lambda}_t} \sum_{m=1}^{M} \left[\boldsymbol{\Lambda}_t' \mathbf{L}(X_t^m, \widetilde{\pi}_t) - H^{-1}(\beta \widehat{V}_{t+1}(\widetilde{X}_{t+1}^m))\right]^2$
    [Approximate conditional expectation] $\widehat{\Phi}_t(X_t, \widetilde{\pi}_t) := H^B(\widehat{\boldsymbol{\Lambda}}_t' \mathbf{L}(X_t, \widetilde{\pi}_t))$
    for $m = 1$ **to** M **do**
      $\widehat{X}_t^m := \widetilde{X}_t^m$
      [Optimal control] $\pi_t^*(\widehat{X}_t^m) := \arg\sup_{\pi_t \in A} \left\{R_t(\widehat{X}_t^m, \pi_t) + \widehat{\Phi}_t(\widehat{X}_t^m, \pi_t)\right\}$
      [Update value function with optimal paths]
      $\widehat{V}_t(\widehat{X}_t^m) := R_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m))$
      $\widehat{X}_{t+1}^m := \mathcal{T}_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m), z_t^m)$
      for $t_j = t + 1$ **to** $N - 1$ **do**
        $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{t_j - t} R_{t_j}(\widehat{X}_{t_j}^m, \pi_{t_j}^*(\widehat{X}_{t_j}^m))$
        $\widehat{X}_{t_j+1}^m := \mathcal{T}_t(\widehat{X}_{t_j}^m, \pi_{t_j}^*(\widehat{X}_{t_j}^m), z_{t_j}^m)$
      end for
      $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{N-t} R_N(\widehat{X}_N^m)$
    end for
  end if
end for

Regression surface versus realised value

Kharroubi et al. (2014) present two alternative versions of the control randomisation algorithm: the one that uses the regression surface to update the value function,

$$\widehat{V}_t(X_t) = R_t(X_t, \pi_t^*(X_t)) + \widehat{\Phi}_t(X_t, \pi_t^*(X_t))$$

and another one that uses the realised value function,

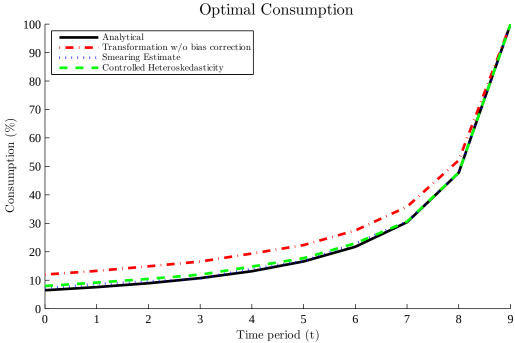
$$\widehat{V}_t(X_t) = R_t(X_t, \pi_t^*(X_t)) + \beta \widehat{V}_{t+1}(X_{t+1})$$

The first algorithm is the so-called value function iteration (VFI), while the second one is the so-called policy function iteration (PFI). The PFI requires recalculation of the sample paths for $t+1$ to T after each iteration backwards in time, as the optimal control affects the future state variables hence changes the simulated paths.

Example - CRRA Consumption model

Consider a simple multi-period utility model where $R_t(X_t, \alpha_t) = \frac{1}{\gamma}(X_t \alpha_t)^\gamma$ and agent optimises consumption each period, $\alpha_t \in (0, 1)$. Endogenous state variable wealth X_t^π grows between periods based on a stochastic return $Z_t \sim \mathcal{N}(\mu = 0.05, \sigma = 0.2)$:

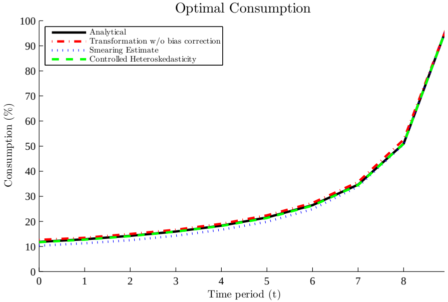
$$X_{t+1} = X_t(1 - \alpha_t)e^{Z_{t+1}}$$

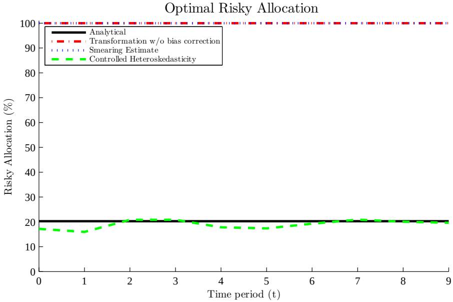


Example - CRRA Consumption and Investment model

Simple multi-period CRRA utility model where $R_t(X_t, \alpha_t) = \frac{1}{\gamma}(X_t \alpha_t)^\gamma$. Agent optimises consumption and risky asset allocation each period, $\pi_t = (\alpha_t, \delta_t) \in [0,1] \times [0,1]$. $\gamma = -10$, endogenous state variable wealth X_t grows between periods based on a stochastic return $Z_t \sim \mathcal{N}(0.1, 0.2)$ and deterministic rate $r = 0.03$:

$$X_{t+1} = X_t(1 - \alpha_t)e^{\delta_t Z_{t+1} + (1-\delta_t)r}.$$





Retirement Model Extensions

Introduce stochastic real interest rate as a Vasicek process. Yearly discretised and simulated with

$$r_{t+1} = \bar{r} + e^{-b}(r_t - \bar{r}) + \sqrt{\frac{\sigma_R^2}{2b}(1 - e^{-2b})}\epsilon_{t+1}, \quad \epsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0,1),$$

where $\bar{r} \in \mathbb{R}^+$ is the long term mean, $b \in (0,1]$ the speed of adjustments and σ_R the volatility.

Introduce a separate taxable deposit account $\widetilde{W}_t$, which is important since the pension account does not allow for deposits in retirement. Always preferred for spending over liquid wealth. Same dynamics and assumptions as liquid wealth.

Model Extension: deposit account

Let ν_t be the minimum withdrawal rate. If $C_t \leq \widetilde{W}_t + P_t + \nu_t W_t$, then

$$W_t^+ = W_t(1 - \nu_t),$$
$$\widetilde{W}_t^+ = \widetilde{W}_t + P_t + \nu_t W_t - C_t,$$

otherwise

$$W_t^+ = W_t + \widetilde{W}_t + P_t - C_t,$$
$$\widetilde{W}_t^+ = 0.$$

where consumption should satisfy the constraint that pension account is non-negative

$$\widetilde{W}_t + W_t + P_t - C_t \geq 0.$$

The remaining liquid wealth after drawdown is invested in a risky asset with real stochastic annual return Z_t and a cash asset growing at the real rate $\tilde{r}_{t,t+1}$.

Evolution of wealth accounts over $(t, t+1)$:

$$W_{t+1} = W_t^+ \left(\delta_t e^{Z_{t+1}} + (1-\delta_t)e^{\tilde{r}_{t,t+1}}\right),$$
$$\widetilde{W}_{t+1} = \widetilde{W}_t^+(\delta_t e^{Z_{t+1}} + (1-\delta_t)e^{\tilde{r}_{t,t+1}}) - \Theta\left(\widetilde{W}_t^+(\delta_t e^{Z_{t+1}} + (1-\delta_t)e^{\tilde{r}_{t,t+1}}) - \widetilde{W}_t^+\right),$$

where function $\Theta(x)$ calculates the tax on the deposit account earnings.

The cash asset annual growing rate is

$$\tilde{r}_{t,t+1} = \int_t^{t+1} r_u du,$$

where the short rate r_t is assumed to follow a Vasicek process

$$dr_t = b(\bar{r} - r_t)dt + \sigma_R dB(t),$$

Model Extension 1 - Annuities

A retiree can at any time $t \in \{t_0, ..., T-1\}$ make a (non-reversible) decision to purchase an annuity for amount A_t that will provide annual life time payments y_t (constant in real terms) starting from $t+1$. This leads to a new state variable Y_t, which holds the information on the size of annuity payments each period evolving as

$$Y_{t+1} = Y_t + y_t,\ Y_{t_0} = 0.$$

The evolution of the pension W_t and deposit $\widetilde{W}_t$ accounts:
If $C_t + A_t \leq \widetilde{W}_t + P_t + \nu_t W_t + Y_t$, then

$$\begin{aligned} W_t^+ &= W_t(1-\nu_t), \\ \widetilde{W}_t^+ &= \widetilde{W}_t + P_t + \nu_t W_t - C_t + Y_t - A_t, \end{aligned}$$

otherwise

$$\begin{aligned} W_t^+ &= W_t + \widetilde{W}_t + P_t - C_t + Y_t - A_t, \\ \widetilde{W}_t^+ &= 0, \end{aligned}$$

To ensure that the pension account W_t is nonnegative, the possible actions C_t and A_t should satisfy the constraint:

$$W_t + \widetilde{W}_t + P_t + Y_t - C_t - A_t \geq 0$$

in addition to $A_t \geq 0,\ C_t > \bar{c}_d$.

Then the optimisation problem should be solved with state vector extended to $X_t = (W_t, \widetilde{W}_t, G_t, H_t, r_t, Y_t)$ to find optimal $\pi_t = (C_t, \delta_t, A_t)$ for $t = t_0, ..., T-1$.

The price of annuity purchased by retiree is

$$a_t(y) := \sum_{i=t+1}^{T} {}_i p_t^{1-h} J(t, i, y),$$

where $J(t, i, y)$ represents the price of an inflation linked zero coupon bond at time t with maturity i and face value y, ${}_i p_t$ is the probability of surviving from year t to i, $h = 0.15$ is price loading. This means that y_t should be found by solving $A_t = a_t(y_t)$.

At time t, the price of a bond with maturity t' is

$$J(t,t',y) = y\mathbb{E}^{\widetilde{Q}}[e^{-\int_t^{t'} r_\tau d\tau}] := ye^{-r(t,t')(t'-t)},$$

where $\widetilde{Q}$ is the risk-neutral probability measure for pricing interest rate derivatives and $r(t,t')$ is the zero rate (yield) from t to t'. The corresponding Vasicek risk-neutral process is

$$dr_t = [b(\bar{r} - r_t) - \lambda\sigma_R]dt + \sigma_R d\tilde{B}(t),$$

where λ is the market price of risk and $\widetilde{B}_t$ is the standard Brownian motion under $\widetilde{Q}$ and

$$r(t,t') = \frac{-\ln A(t,t') + B(t,t')r_t}{t'-t},\ B(t,t') = \frac{1}{b}\left(1 - e^{-b(t'-t)}\right).$$

$$A(t,t') = \exp\left[(B(t,t') - t' + t)\left(\bar{r} - \frac{\lambda\sigma_R}{b} - \frac{\sigma_R^2}{2b^2}\right) - \frac{\sigma_R^2}{4b}B(t,t')^2\right],$$

This means that $a_t(y)$ depends on r_t.

We estimate parameters using two-stage procedure: the real r_t process is estimated using spot interest rate data and then the market price of risk λ is estimated using termstructure of zero coupon bonds.

Annuities in the Age pension means tests in 2017

The definition of annuity income for the income test is

$$y_t - \frac{a_{t_x}(y_t)}{e_x - t_x},$$

where t_x is the annuity purchasing time and e_x is the life expectancy at time t_x.

In the asset test, the value of the annuity is assumed to be equal to the original purchase price of the annuity with a linear yearly value decrease until the life expectancy age is reached, i.e.

$$\max\left(a_{t_x}(y_t) - \frac{a_{t_x}(y_t)}{e_x - t_x}(t - t_x), 0\right).$$

These rules cause some difficulties to our model, as it will require additional state variables in terms of annuity purchase price and annuity purchasing time (which complicates the problem definition further as it is allowed to add on to annuities later in retirement).

Annuities in the Age pension means tests – approximation

Function for income test

$$P_{\mathrm{I}} := P^{d}_{\max} - \left(P_{\mathrm{D}}(W_t) + Y_t(1-\Upsilon) - L^{d}_{\mathrm{I}} \right) \varpi^{d}_{\mathrm{I}}.$$

$\Upsilon = 0.9.$

Function for the asset-test

$$P_{\mathrm{A}} := P^{d}_{\max} - \left(W_t + a_t(Y_t) - L^{d,h}_{\mathrm{A}} \right) \varpi^{d}_{\mathrm{A}}.$$

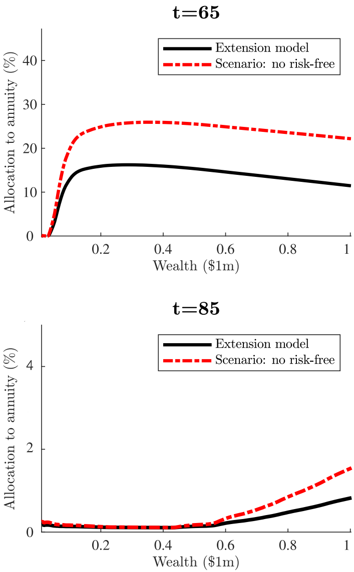


Figure: Optimal annuitisation at age t versus liquid wealth (no prior annuitisation).

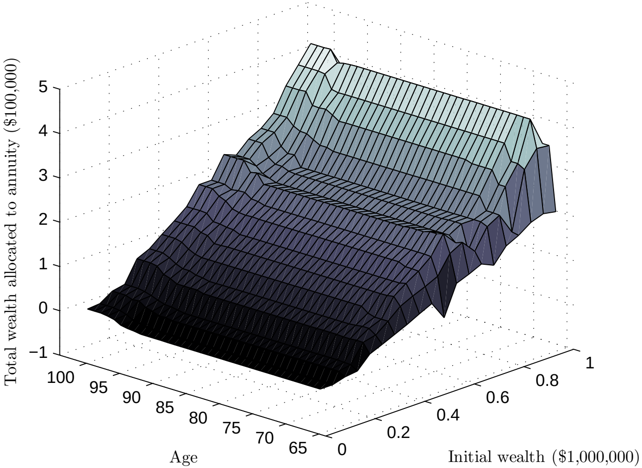


Figure: Optimal cumulative annuitisation over time given initial liquid wealth.

Conclusions - Extension 1

- It is optimal to annuitise earlier rather than later in retirement (due to the mortality credit). The exception is for very poor households.
- Delaying annuitisation leads to less wealth annuitised, but higher annuity payments.
- The means-test decreases the 'demand' for annuities, but does not eliminate it. Retiree with low likelihood to access Age Pension has constant annuitisation rate.
- The mortality credit from the annuity dominates the utility received from bequeathing this wealth. **Optimal annuitisation is the same with/out access to a risk-free rate** when loading on annuity premium is zero.

Model Extension 2 - Flexible housing

Australian retirees are ‘house rich, but asset poor’, and can optimise Age Pension payments by overallocating to the family home. We extend the model to flexible housing decisions by scaling housing and access to a reverse mortgage.

Reverse mortgage:

- Loan against the home equity up to an age dependent loan-to-value ratio, with no amortisation/interest payments required.
- Starts at 20-25% at age 65, and increases 1% per year.
- Multiple options how to access: lump sum, credit line, tenure, etc.
- Interest and fees accumulate, but is capped by the house value.
- At death (or sale of home) the loan is paid off, and any equity remaining is returned.

Model Extension 2 - Flexible housing

- The retiree can at any time up or downscale housing with a proportion $\tau_t \in [-1, \infty]$, to get a new house valued

$$H_{t+1} = H_t(1 + \tau_t).$$

- **If $\tau_t \neq 0$, transaction cost applies to the current house value.**
- The retiree can at any time chose a proportion $l_t \in [0, l(t)]$ up to loan-to-value threshold $\overline{L}_t$ as a reverse mortgage from the home value, which adds to the outstanding loan state L_t.
- The loan-to-value ratio is time dependent and defined as a proportion of the home value $\overline{L}_t = H_t l(t)$ where

$$l(t) = 0.2 + 0.01(\min(85, t) - 65).$$

- The loan value state variable therefore evolves as

$$L_{t+1} = (L_t \mathbb{I}_{\{\tau_t = 0\}} + l_t H_t(1 + \tau_t))e^{\widetilde{r}_{t,t+1} + \varphi},$$

Model Extension 2 - Flexible housing

- The costs of any decision (transaction cost, the difference in house assets in case of scaling and repayment of loan) is reflected in the wealth process. Define

$$b(l_t, \tau_t, L_t, H_t) := l_t H_t(1+\tau_t) - \mathbb{I}_{\tau_t \neq 0}\left(H_t(\tau_t+\eta)+L_t\right)$$

  to represent all changes to the wealth from house scaling and reverse mortgage decisions, where η is the proportional transaction cost.
- The evolution of the pension W_t and deposit $\widetilde{W}_t$ accounts:
  If $C_t \leq \widetilde{W}_t + P_t + \nu_t W_t + b(l_t, \tau_t, L_t, H_t)$, then

$$\begin{aligned} W_t^+ &= W_t(1-\nu_t), \\ \widetilde{W}_t^+ &= \widetilde{W}_t + P_t + \nu_t W_t + b(l_t, \tau_t, L_t, H_t) - C_t, \end{aligned}$$

  otherwise

$$\begin{aligned} W_t^+ &= W_t + \widetilde{W}_t + P_t + b(l_t, \tau_t, L_t, H_t) - C_t, \\ \widetilde{W}_t^+ &= 0. \end{aligned}$$

Model Extension 2 - Flexible housing

The bequest function needs to include the house asset after any reverse mortgage has been repaid, and becomes $U_B(W_t + \widetilde{W}_t, \max(H_t - L_t, 0))$.

Then the optimisation problem should be solved with the state vector extended to $X_t = (W_t, \widetilde{W}_t, G_t, H_t, r_t, L_t)$ to find optimal $\pi_t = (C_t, \delta_t, \tau_t, l_t)$ for $t = t_0, ..., T-1$.

We set cost of selling house: $\eta = 6\%$, interest rate markup: $\varphi = 0.0242$; and risk-free rate parameters: $b = 0.64$, $\bar{r} = 0.013$ and $\sigma_R = 0.016$.

Some constraints need to be imposed on the control variables.

- The option to take out (or add to) a reverse mortgage is bounded from above by the difference of any outstanding mortgage and the LVR, hence

$$l_t \leq \max\left(0, \frac{\overline{L}_t - L_t \mathbb{I}_{\tau_t=0}}{H_t(1+\tau_t)}\right).$$

  Note that if the control variable τ_t for scaling housing is not 0, any outstanding reverse mortgage must be paid back in full and a new reverse mortgage is available against the new house value.

- For the scaling of housing, an upper bound for how much the house asset can be increased is

$$\tau_t \leq \frac{W_t + \widetilde{W}_t - \mathbb{I}_{\tau\neq 0}\left(\eta H_t + L_t\right)}{H_t}.$$

  The lower bound is simply -1.

- Finally, the budget constraint should be satisfied

$$b(l_t, \tau_t, L_t, H_t) + W_t + \widetilde{W}_t + P_t - C_t \geq 0$$

  The constraint ensures that the wealth is enough to cover consumption and scaling housing/reverse mortgage costs.

Results - Extension 2

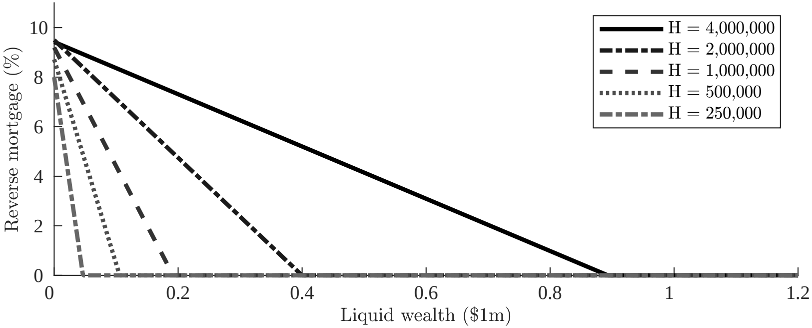
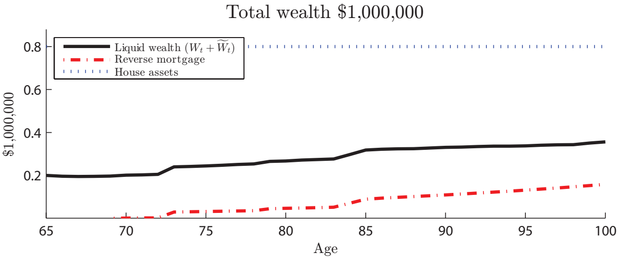


Figure: Optimal proportion reverse mortgage at the retirement age.



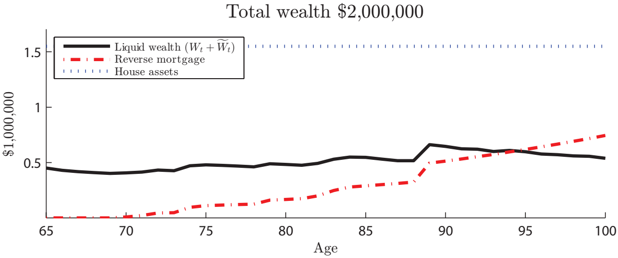


Figure: Wealth, house and reverse mortgage paths in retirement given initial total wealth.

Conclusions - Extension 2

- The proportion reverse mortgage increases with house value and decreases with wealth (confirms empirical results in Chiang and Tsai (2016)).
- The proportion increases with age; never reaches the LVR.
- **It is never optimal to downsize housing, even when overallocated, unless certain events are incurring significant costs.**
- Scaling housing is more costly than reverse mortgages for accessing part of home equity. In addition, a reverse mortgage allows the retiree to still receive utility from the larger home.
- Only marginal effect on initial housing allocation with the additional control variables.

Thank you for your attention!

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