Insider trading with penalties in continuous time

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Outline of talk

The Kyle model without frictions

Kyle-Back model of informed trading

Inspired by Kyle (1985), Back (1992) studies a market for a bond and a risky asset with three types of participants:

1. *Noise traders:* The noise traders have no private information and are not strategic. Their cumulative demand is given by σB, where B is a Brownian motion and σ is constant.
2. *Informed trader, a.k.a. insider:* The insider knows the value of the risky asset at time 1, which is given by a random variable, V, independent of B. Being risk-neutral, her objective is to maximize her expected profit.
3. *Market makers:* The market makers observe the total order and set the price of the risky asset to clear the market via a Bertrand competition.

The pricing mechanism of the market

- Market makers decide the price by looking at the total order

$$Y_t = \sigma B_t + \theta_t,$$

where θ_t is the position of the insider in the risky asset at time t.

- Thus, the filtration of the market maker is the one generated by Y. Note that θ is not necessarily adapted to $\mathcal{F}^Y$.
- The market makers have a *pricing rule*, $H : [0,1] \times \mathbb{R} \mapsto \mathbb{R}$, to assign the price in the following form:

$$S_t = H(t, Y_t),$$

where S_t is the market price of the risky asset at time t. H is strictly increasing in Y.

Equilibrium

- The market makers choose a *rational* pricing rule, i.e. a pricing rule so that

$$H(t, Y_t) = \mathbb{E}[V|\mathcal{F}_t^Y],$$

for every $t \in [0, 1]$.
- The insider aims to maximize her expected profit out of trading.
- Equilibrium is a pair (H^*, θ^*) such that the following conditions are satisfied:
  1. *Market efficiency:* Given θ^*, H^* is a rational pricing rule.
  2. *Insider optimality:* Given H^*, θ^* maximizes the expected profit of the insider.

Kyle model with penalties

Insider trading with legal penalties

- Recently Carré, Collin-Dufresne and Gabriel (2022, JET) and Kacperczyk and Pagnotta (2023, forthcoming in JoF) study in a one-period model the Kyle equilibrium when the insider is subject to additional transaction costs (or legal risk).

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- These additional costs could be a result of frictions in executing large portfolios, or
- The penalties arise in case the informed trader is an illegal insider and pays a penalty (in addition to losing all her profits) after investigation.
- Carré et al. consider general convex penalties and uniformly distributed noise trades while Kacperczyk and Pagnotta have quadratic penalties and normally distributed noise demand.

Penalties in continuous time

- Let's associate the following quadratic transaction cost by time t to the strategy $d\theta_t = \alpha_t dt$:

$$C_t := \frac{c}{2} \int_0^t \alpha_s^2 ds.$$

  for some $c > 0$.

- Her objective is still to maximize the expected final wealth, W_1, given by

$$W_1 = \int_0^1 (V - S_s)\alpha_s ds - \frac{c}{2} \int_0^1 \alpha_s^2 ds.$$

Legal penalties in continuous time

- The above cost structure can also arise as a legal penalty.
- Indeed, suppose an investigation identifies illegal inside trading with probability p, after which the insider pays a legal penalty of $k \int_0^1 \alpha_t^2 dt$.

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- Indeed, suppose an investigation identifies illegal inside trading with probability p, after which the insider pays a legal penalty of $k \int_0^1 \alpha_t^2 dt$.
- The expected profit of the insider under this scenario is

$$E^v\left[(1-p)\int_0^1 (V - S_s)\alpha_s ds - pk\int_0^1 \alpha_s^2 ds\right]$$
$$= (1-p)E^v\left[\int_0^1 (V - S_s)\alpha_s ds - \frac{pk}{1-p}\int_0^1 \alpha_s^2 ds\right].$$

Thus the coefficient c from the previous slide can be associated with $\frac{pk}{1-p}$, which gets large as the probability of a successful investigation gets bigger.

Insider's optimality

Recall $dY_t = \sigma dB_t + \alpha_t dt$, and let

$$J(t,y) = \sup_{\alpha \in \mathcal{A}(H)} E^v \left[\int_t^1 (v - H(u, Y_u))\alpha_u du - \frac{c}{2} \int_t^1 \alpha_t^2 dt \middle| Y_t = y \right].$$

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Direct calculations lead to

$$J_t + \frac{\sigma^2}{2} J_{yy} + \sup_{\alpha} \left\{ \alpha(J_y + v - H) - \frac{c\alpha^2}{2} \right\} = 0.$$

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Therefore,

$$\alpha^* = \frac{J_y(t,y) + v - H(t,y)}{c}. \tag{1}$$

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and

$$J_t + \frac{\sigma^2}{2} J_{yy} + \frac{(J_y + v - H)^2}{2c} = 0. \tag{2}$$

A quadratic BSDE

Next suppose there exists a smooth function J^0 such that

$$J^0_t + \frac{\sigma^2}{2} J^0_{yy} = 0, \qquad J^0_y = H - v. \tag{3}$$

Thus, if one defines $u = J - J^0$ and conjectures that $J(1, \cdot) \equiv 0$, one obtains

$$u_t + \frac{1}{2}\sigma^2 u_{yy} + \frac{1}{2c} u_y^2 = 0, \quad u(1, y) = -j^0(y, v) := -J^0(1, y). \tag{4}$$

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This has a simple (BSDE) formulation

$$dU_t = \sigma Z_t dB_t - \frac{1}{2c} Z_t^2 dt, \quad U_1 = u(1, \sigma B_1), \tag{5}$$

whose solution is given by $u(t, x) = c\sigma^2 \log \rho(t, x, v)$, where

$$\rho(t, x, v) := E^v \left[\exp\left(-\frac{j^0(\sigma B_1, v)}{c\sigma^2} \right) \Big| \sigma B_t = x \right]. \tag{6}$$

Towards a recipe for equilibrium

- The above implies the optimal control of the insider is $\alpha^*(t, Y_t, V)$, where

$$\alpha^*(t, y, v) := \sigma^2 \frac{\rho_y(t, y, v)}{\rho(t, y, v)}, \tag{7}$$

implying an *h*-transformation in making!

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$$\mathbb{E}\left[\frac{\rho_y(t, Y_t, V)}{\rho(t, Y_t, V)}\middle|\mathcal{F}_t^Y\right] = \int \rho_y(t, Y_t, v)\Pi(dv)$$

$$= \frac{d}{dy}\int \rho(t, y, v)\Pi(dv)\Big|_{y=Y_t} = 0,$$

Filtering problem

Proposition 1

Suppose that there exists a continuous function j^0 such that $\rho(0,0,\cdot) \equiv 1$, where ρ is defined via (6). Assume further that there exists a unique strong solution on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\in[0,1]}, \mathbb{Q})$ to

$$Y_t = \sigma B_t + \int_0^t \sigma^2 \frac{\rho_y(s, Y_s, V)}{\rho(s, Y_s, V)} ds$$

such that

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^t \Big(\frac{\rho_y(s, Y_s, V)}{\rho(s, Y_s, V)}\Big)^2\Big] < \infty, \quad t \in [0,1]. \tag{8}$$

Then,

$$\rho(t, Y_t, v)\Pi(dv) = \mathbb{P}(V \in dv | \mathcal{F}_t^Y), \qquad t \in [0,1].$$

A recipe for equilibrium

1. Find a continuous function j^0 such that i) $\rho(0, 0, \cdot) \equiv 1$, where ρ is given by (6), and ii) it is differentiable in its first parameter with $j^0_y(y, v) = h(y) - v$. Note that the first condition entails

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2}\right) \exp\left(-\frac{j^0(y, v)}{\sigma^2 c}\right) dy = 1, \qquad \forall v. \tag{9}$$

2. Set

$$H_t + \frac{1}{2}\sigma^2 H_{yy} = 0, \qquad H(1, y) = h(y).$$

3. Show that (H, θ^*) with $d\theta^*_t = \sigma^2 \frac{\rho_y(t, Y_t, V)}{\rho(t, Y_t, V)} dt$ is equilibrium provided they are admissible by using the candidate value function $J = J^0 + u$ that satisfies (2).

Emergence of Schrödinger potentials

Towards a fixed-point algorithm

- **Notation:** $\hat{c} := c\sigma^2$. That $j^0_y = h(y) - v$ for some h to be determined implies we are searching for a j^0 such that

$$j^0(y, v) = \Psi(v) + \phi(y) - yv. \tag{10}$$

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- Next, since $\exp(-j^0(y,v)/\hat{c})\Pi(dv)$ is the conditional distribution of V given $Y_1 = y$, it must integrate to 1:

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$$\int_{\mathbb{R}} \exp\left(\frac{yv - \phi(y)}{\hat{c}}\right)p(\sigma, y)dy = \exp\left(\frac{\Psi(v)}{\hat{c}}\right),$$

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- Thus, if they exist, $\phi/\hat{c}$ and $\Psi/\hat{c}$ are the *unique* potentials of the entropic optimal transport problem from $N(0, \sigma^2)$ to Π with the quadratic "cost function" $\frac{1}{2\hat{c}}(x - y)^2$.

Existence of solutions

- Given the positivity of the cost function, the associated entropic optimal transport problem has a solution (see lecture notes by M. Nutz) with the potentials ϕ and Ψ solving the *Schrödinger equations*

$$\begin{aligned} \phi(y) &= \hat{c}\log\int_{f(\mathbb{R})}\exp\left(\frac{yv-\Psi(v)}{\hat{c}}\right)\Pi(dv), \\ \Psi(v) &= \hat{c}\log\int_{\mathbb{R}}\exp\left(\frac{yv-\phi(y)}{\hat{c}}\right)p(\sigma,y)dy. \end{aligned} \tag{11}$$

- Moreover, $\mathbb{E}[|\phi(\sigma B_1)| + |\Psi(V)|] < \infty$.

Differentiability of solutions

- Assume that V has all exponential moments. Then it is a simple exercise to show that ϕ and Ψ are infinitely differentiable.
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- In particular,

$$\phi'(y) = \int v\nu_\Psi(dv),\ \phi''(y) = \frac{1}{\hat{c}}\Big(\int v^2\nu_\Psi(dv) - \Big(\int v\nu_\Psi(dv)\Big)^2\Big), \tag{12}$$

where

$$\nu_\Psi(dv) = \frac{\exp\left(\frac{yv-\Psi(v)}{\hat{c}}\right)\Pi(dv)}{\int \exp\left(\frac{yv-\Psi(v)}{\hat{c}}\right)\Pi(dv)}.$$

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- Note: ν_Ψ will be the conditinal distribution of V given Y_1 in equilibrium!

Gaussian case

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- $\phi^*(y) = \frac{\lambda^* y^2}{2} + \mu y$ for

$$\lambda^* = \frac{-c + \sqrt{c^2 + 4\frac{\gamma^2}{\sigma^2}}}{2}. \tag{13}$$

Moreover,

$$\Psi^*(v) = \frac{\hat{c}}{2} \log \frac{c}{c + \lambda^*} + \frac{(\mu - v)^2}{2(c + \lambda^*)}. \tag{14}$$

Equilibrium for the Kyle-Back model with penalties

Equilibrium

- Now, denote the solution of (11) by (ϕ^*, Ψ^*) with the normalisation that $\phi^*(0) = 0$.
- Next define

$$H^*(t,y) = \int \rho^*(t,y,z) z \Pi(dz), \tag{15}$$

where

$$\rho^*(t,y,v) := E^v\left[\exp\left(-\frac{j^*(\sigma B_1, v)}{\hat{c}}\right)\Big|\sigma B_t = y\right]. \tag{16}$$

- Observe that

$$H_t^* + \frac{\sigma^2}{2} H_{yy}^* = 0, \qquad H^*(1,\cdot) = h^* = \frac{d\phi^*}{dy}. \tag{17}$$

Equilibrium

- Then (H^*, θ^*) is an equilibrium where

$$d\theta_t^* = \sigma^2 \frac{\rho_y^*(t, Y_t, V)}{\rho^*(t, Y_t, V)} dt, t \in [0, 1], \text{ and } \theta_0^* = 0. \qquad (18)$$

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$$E^{0,v}[W_1^{\theta^*}] = J(0,0) = \Psi^*(v) + E^{0,v}[\phi^*(\sigma B_1)]. \qquad (19)$$

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- The *price inefficiency* of the equilibrium, denoted by δ, is given by

$$\delta := \mathbb{E}[\mathrm{Var}(V|\mathcal{F}_1^{Y^*})] = \hat{c}\mathbb{E}\left[\frac{d^2\phi^*}{dy^2}(Y_1^*)\right]. \tag{20}$$

Alternative representation of the optimal strategy

- Suppose $H^*(1,\cdot)$ is of at most exponential growth. Then,

$$\alpha_t^* := \frac{d\theta_t^*}{dt} = \frac{1}{c}(v - E^{0,v}[h^*(Y_1^*)|\mathcal{F}_t^I]), \tag{21}$$

with $E^{0,v}[h(Y_1^*)|\mathcal{F}_t^I] = \mathcal{P}(t, Y_t^*; v)$, where

$$\mathcal{P}(t,y;v) = \frac{\int_{\mathbb{R}} h^*(x)\exp(\frac{vx-\phi^*(x)}{\hat{c}})p(\sigma\sqrt{1-t}, x-y))dx}{\int_{\mathbb{R}} \exp(\frac{vx-\phi^*(x)}{\hat{c}})p(\sigma\sqrt{1-t}, x-y))dx}.$$

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- Thus, insider trades on the differential between her private signal and the expected terminal price, and trades aggressively if the penalty, i.e. c, is small.

Connection with h-transforms

- Consider the SDE associated to the equilibrium demand:

$$Y_t^* = \sigma B_t + \sigma^2 \int_0^t \frac{\rho_y^*(s, Y_s^*, v)}{\rho^*(s, Y_s, v)} ds. \tag{22}$$

- Let $\mathbb{Q}^*$ be the law induced by Y^* on the space of continuous functions on $[0, 1]$ vanishing at 0.

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- Let $(\mathcal{B}_t)_{t \in [0,1]}$ be the right continuous augmentation of the natural filtration of the coordinate process X, and $\mathbb{W}$ be the Wiener measure.

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- Let $(\mathcal{B}_t)_{t\in[0,1]}$ be the right continuous augmentation of the natural filtration of the coordinate process X, and $\mathbb{W}$ be the Wiener measure.
- Then $\mathbb{Q}^*$ is given by the following h-transform of Brownian motion:

$$\mathbb{E}^{\mathbb{Q}^*}[F] = \frac{\mathbb{E}^{\mathbb{W}}\left[F \exp\left(\frac{v\sigma X_1 - \phi^*(\sigma X_1)}{\hat{c}}\right)\right]}{\mathbb{E}^{\mathbb{W}}\left[\exp\left(\frac{v\sigma X_1 - \phi^*(\sigma X_1)}{\hat{c}}\right)\right]}, \quad F \in \mathcal{B}_1. \qquad (23)$$

Relative entropy of the h-transform

- The relative entropy of the above change of measure is not only finite but can be computed explicitly.
- Indeed,

$$H(\mathbb{Q}^*||\mathbb{W}) = \frac{v(\Psi^*)'(v) - \Psi^*(v) - E^{0,v}[\phi^*(Y_1^*)]}{\hat{c}}$$

- Consequently, the insider expects to the following penalty in equilibrium:

$$v(\Psi^*)'(v) - \Psi^*(v) - E^{0,v}[\phi^*(Y_1^*)].$$

Back to the Gaussian case

- For a better parametrisation suppose $c = \kappa\frac{\gamma}{\sigma}$.
- Then

$$dY_t^* = \sigma dB_t + \Lambda(\kappa)\sigma\frac{\frac{V-\mu}{\gamma} - \frac{\Lambda(\kappa)Y_t^*}{\sigma}}{1 - t\Lambda^2(\kappa)}dt,$$

  where $\Lambda(\kappa) = \frac{\sqrt{\kappa^2+4}-\kappa}{2}$.
- $\lambda^* = \Lambda(\kappa)\frac{\gamma}{\sigma}$.

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where $\Lambda(\kappa) = \frac{\sqrt{\kappa^2+4}-\kappa}{2}$.
- $\lambda^* = \Lambda(\kappa)\frac{\gamma}{\sigma}$.
- Y^* is a martingale for market makers, but it is an O-U process mean reverting around $\frac{(V-\mu)\sigma}{\gamma\Lambda(\kappa)}$ for the insider.

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- Y^* is a martingale for market makers, but it is an O-U process mean reverting around $\frac{(V-\mu)\sigma}{\gamma\Lambda(\kappa)}$ for the insider.
- As $\kappa \to 0$, Y^* converges to

$$dY_t^* = \sigma dB_t + \sigma \frac{\frac{V-\mu}{\gamma} - \frac{Y_t^*}{\sigma}}{1 - t} dt;$$

  that is, $\frac{Y^*}{\sigma}$ becomes a Brownian bridge from 0 to $\frac{V-\mu}{\gamma}$.

Asymptotics for large penalties

- In the above Gaussian setting

$$\alpha_t^* \sim N\Big(\frac{V-\mu}{\gamma}\Lambda(\kappa)\sigma, \frac{t\sigma^2\Lambda^4(\kappa)}{1-t\Lambda^2(\kappa)}\Big).$$

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- As $\kappa \to \infty$, $\Lambda \to 0$. Thus, even after normalizing the insider's trades by $\Lambda(\kappa)$, the standard deviation of the trading rate remains small, that is, order of $\Lambda(\kappa)$.
- Thus, the insider buys (sells) at the constant rate $\frac{|V-\mu|}{\gamma}\Lambda(\kappa)\sigma$ if her private value is greater (smaller) than the initial price, μ, of the asset.

Expected penalties are non-monotone

Insider's expected wealth is decreasing in κ. However, the expected penalty is not monotone!

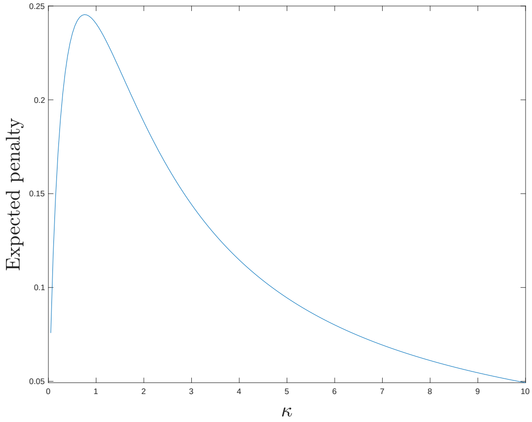

Figure: Expected penalties in equilibrium normalised by $\gamma\sigma$.

Regulating insider trading

Regulator's dilemma

- Suppose V is Gaussian, and the regulator has the following simple objective:

$$\begin{aligned} &\min_{\kappa} \Lambda(\kappa)\gamma\sigma + R\kappa\Lambda(\kappa)\gamma^2, \\ \text{subject to } & -\frac{\gamma\sigma}{2}\kappa \log(\kappa\Lambda(\kappa)) \geq b, \end{aligned} \tag{24}$$

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- Note that the first term is the expected loss of the noise traders and the second is the expected post-trade variance adjusted by the factor R, which measures the sensitivity of the policy toward price efficiency.
- Since the expected penalties are bounded, if

$$b^0 := \frac{b}{\gamma\sigma} < \bar{P} := \sup -\frac{1}{2}\kappa \log(\kappa\Lambda(\kappa)),$$

  the budget constraint cannot be satisfied. Then it can be shown that if the amount of noise trading is rather small, the regulator does not run an investigation.

Optimal policy

- Now suppose $b^0 \geq \bar{P}$. Define

$$\mathcal{K}(a) := \{\kappa : -\kappa \log(\kappa \Lambda(\kappa)) \geq a\}. \tag{25}$$

- Then the optimal penalty rate is $c^* := \kappa^* \frac{\gamma}{\sigma}$, where

$$\kappa^* = \arg\min_{\kappa \in \mathcal{K}(b_0)} \Lambda(\kappa) + R\frac{\gamma}{\sigma}(1 - \Lambda^2(\kappa)). \tag{26}$$

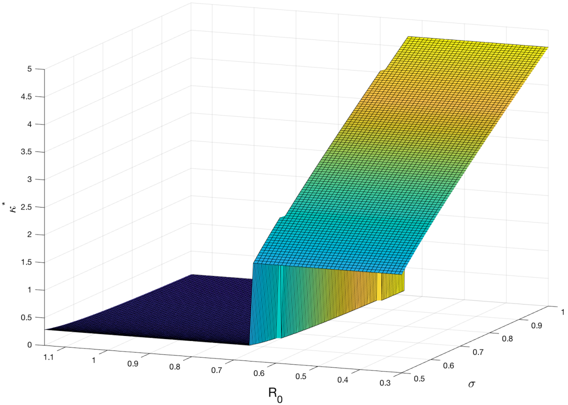

Figure: The optimal penalty rate κ^* as a function of noise volatility σ, and the regulator's sensitivity toward price efficiency, $R_0 = R\frac{\gamma}{\sigma}$. The figure assumes $\gamma = 1$ and $b = 0.1$.

Conclusion

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- The solution reveals an interesting connection between quadratic BSDEs and h-transforms, where the terminal condition of the BSDE is determined in equilibrium.
- One can use this setup to solve the regulators's problem with the objective to minimise uninformed traders losses but also keep the informational efficiency above a certain level. For details, see
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- Extension to general convex cost functions will be of great interest as it is important for the regulator to decide the best cost functional to regulate insider trading.